THE TERNARY OPERATION $(abc)=ab^{-1}c$ OF A GROUP

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1. Introduction. This note is the result of some investigations into the ternary operation $ab^{-1}c$ in a group. We shall assume familiarity on the part of the reader with the notions of a group, a one-one transformation (we shall use the shorter term permutation) of an arbitrary set of elements, an automorphism, and a coset.[1] We shall use the multiplicative notation for a group G with elements $a, b, c, \cdots$. We shall also use the following convention for multiplication of permutations. Given two permutations T_i: $x \to xT_i$ $(i=1, 2)$, then T_1T_2 is $x \to (xT_1)T_2$. Finally, we denote automorphisms by small Greek letters.

In §2 we shall review certain properties of the ternary operation in a given group, determining all subsets closed with respect to this operation and the group of permutations of G which preserve this operation. These results had been previously obtained by Reinhold Baer.[2]

In §§3 and 4 we give postulates for this operation with proofs of their independence and consistency. Thus, if a ternary operation satisfies these postulates in an arbitrary set of elements, then the set may be made into a group (unique within isomorphism) in which $(abc)=ab^{-1}c$. The first set of postulates appears as a weakened form of a set given by Baer in his paper,[3] in which he mentions the group property. This and an equivalent set completely determine the ternary function as $ab^{-1}c$. However, by further weakening one of these postulates, it is possible to get a system which no longer has this last property. That is, the group property still holds but the ternary operation is not determined by the group operation.

In the remaining sections we get a geometric interpretation of the ternary operation and derive therefrom simple conditions on pairs of elements (vectors) under which they form a group. In the case where an abelian group is desired, the conditions are even simpler, reducing essentially to a single law.

I wish to express my gratitude to Garrett Birkhoff for his kind

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[1] Cf. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig and Berlin, 1937, pp. 1, 5, 41 and 10.

[2] *Zur Einführung des Scharbegriffs*, J. Reine Angew. Math. vol. 160 (1929) pp. 199–206.

[3] Cf. Baer, op. cit. p. 202, footnote.

assistance and encouragement, without which this note would probably not have been written.

2. The ternary operation in a group.

THEOREM 1. *$S \subset G$ is closed under (abc) if and only if S is a coset of some subgroup of G; indeed a right- (left-) coset of $S^{-1}S$ (SS^{-1}).*

PROOF. For the first part see Baer's paper.[4] As for the second observe that if $S = sT$ ($s \in S$), then $T = s^{-1}S$ ($s \in S$) and hence $T = S^{-1}S$. Similarly, $S = Ts$ ($s \in S$) implies $T = SS^{-1}$.

DEFINITION 1. *The set of all permutations of G of the form αT_a: $x \to (x\alpha)a$, where α is an automorphism of G, is called the holomorph of G, or simply the holomorph.*[5]

THEOREM 2. *The group of all permutations which preserve the ternary operation is the holomorph.*[6]

Explanation. A permutation T preserves the ternary operation, by definition, if and only if $(abc)T = (aT\ bT\ cT)$. The group property follows from the general theorem that the set of all automorphisms of any algebra form a group, and the set in question is exactly that of the automorphisms with respect to the ternary operation.[7]

3. Postulates for the ternary operation. The reader will observe, as stated in the introduction, that we may consider the postulates given below as postulates for a group under the ternary operation. The first set is interesting, considered as postulates for a group, because it does not (explicitly) require the existence of either the identity or the inverse.[8] The other sets require only the existence of an identity. An analogous situation is that of generalized groups defined by the use of an n-ary function.[9] However, the postulates given below seem to be the simplest for the general case.

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[4] Cf. Baer, op. cit. Satz 3 (part 3). As may be seen, Baer's "schar" is simply a coset studied under the operation $ab^{-1}c$. By Theorem 1, and also by the fact that $sT = sTs^{-1}s$, where sTs^{-1} is a subgroup if and only if T is (and indeed equals T if and only if T is normalized by s), we see that the property of being a coset is intrinsic.

[5] Cf. Zassenhaus, op. cit. p. 46.

[6] Cf. Baer, op. cit. Satz 11 (part 3). An alternative proof would be to consider T as given and define $d = eT$, e the identity of G, and α: $x \to (xT)d^{-1}$. Using $ab = (aeb)$, it is easy to verify that $(ab)\alpha = (a\alpha)(b\alpha)$, and that $T = \alpha T_d$. The converse is straightforward.

[7] Cf. Garrett Birkhoff, Proc. Cambridge Philos. Soc. vol. 31 (1935) p. 434.

[8] Cf. Rainich, *Note on group postulates*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 81–84.

[9] Cf. E. L. Post, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 208–350.

We shall assume, unless otherwise stated, that the systems defined below are closed with respect to (abc) and that they contain all elements under discussion.

DEFINITION 2. *Let G be a set of elements on which there is defined a ternary operation (abc) satisfying the following postulates:*

$$(3.1) \qquad A_1\colon ((abc)de) = (ab(cde)), \quad A_2\colon (abb) = a, \quad A_3\colon (bba) = a.$$

We shall call G an abstract coset.

We shall not use these postulates directly but use a weakened yet equivalent set given below. The equivalence is a corollary to Theorem 3 proved below.

$$(3.2) \qquad \begin{aligned} &A_1\colon ((abc)de) = (ab(cde)).\\ &B\colon \text{There exists } u \text{ in the set satisfying (a) } (auu)=a, \text{ (b) } (aau)=u. \end{aligned}$$

THEOREM 3. *If G is a set satisfying (3.2) and we define $ab=(aub)$, then G becomes a group and $(abc)=ab^{-1}c$.*

PROOF. Closure is obvious.

In A_1 take $b=d=u$, and we get the associative law $(ac)e=a(ce)$.

By definition and B(a), it follows that u is a right identity.

If a is given, choose $x=(uau)$. Then $ax=(au(uau))=((auu)au)=(aau)=u$. It follows that G is a group under the binary operation and hence u is a left identity also, that is, $(uua)=ua=a$ for all a.

Finally, $(abc)=((auu)bc)=(au(ubc))=a(ubc)=a(ub(uuc))=a((ubu)uc)=ab^{-1}c$.

COROLLARY 1. *(3.1) and (3.2) are equivalent.*

PROOF. The proof is obvious.

Thus we see that if G satisfies (3.1), we may choose any element u in G and define a group G_u with u as its identity, and $ab=(aub)$ as its law of composition. However, the following corollary shows that we get essentially the same group no matter which element we choose for the identity.

COROLLARY 2. *The groups G_u are isomorphic for all u in G, an abstract coset. Moreover, $(abc)=ab^{-1}c$.*

PROOF. Consider G_u and G_v. Define $T\colon x\to(xuv)$. T is in the holomorph of G_u with α the identity permutation. By Theorem 2 it follows that $(aub)T=(aT\ uT\ bT)$. But $uT=v$, which completes the proof.

Remark. G may thus be considered either as a group or as an abstract coset. We could define the holomorph of an abstract coset as the group of all permutations preserving the ternary operation (abc). This evidently coincides with the holomorph of G (considered as a group) given by Definition 1.

Now we shall examine the system we get from (3.2) by weakening postulate A_1.

DEFINITION 3. *Let G be a set of elements on which there is defined a ternary operation satisfying, for some u in G,*

$$(3.3)\qquad \begin{aligned} &\text{A: } ((auc)de) = (au(cde)),\\ &\text{B: (a) } (auu) = a, \quad \text{(b) } (aau) = u. \end{aligned}$$

THEOREM 4. *If G is a set satisfying Definition 3, and we define $ab = (aub)$, then G becomes a group and the following properties are equivalent:*

$$\text{T: } (abc) = ab^{-1}c, \qquad A_1\text{: } ((abc)de) = (ab(cde)), \qquad A_3\text{: } (bba) = a.$$

PROOF. The fact that G is a group follows from the proof of Theorem 3. It is obvious that T implies A_1 and A_3. But A_1 implies T by Theorem 3. It suffices to prove that A_3 implies T. But, by the proof of Theorem 3 again, $(abc) = a(ubc)$. Now $b(ubc) = (bbc) = c$ or $(ubc) = b^{-1}c$, so the result follows.

The following example shows that T does not hold for all sets satisfying (3.3).

Example. Let G be the set of two elements u and a combined according to the following rules:

$$(uuu) = (aau) = (aaa) = (aua) = u,$$
$$(auu) = (uau) = (uua) = (uaa) = a.$$

Obviously B is satisfied. The reader may check A, noting that it is unnecessary to check A for the first element equal to u since $(uux) = x$ for all x in G. But, since $(aaa) = u \neq a$, we see that (xyz) is not in general equal to $xy^{-1}z$. Also, since any group of order 2 (and there is only one) satisfies Definition 3 with $(abc) = ab^{-1}c$, we see that this set of postulates is insufficient to determine the ternary operation in terms of the group operation.

Remark. It is obvious that if we replace B(b) by A_3 in (3.3) we get a set equivalent to (3.1) and (3.2).

DEFINITION 4. *G is said to be commutative if $(abc) = (cba)$.*

THEOREM 5. *A necessary and sufficient condition that an abstract*

coset be commutative is that $(abc) = (cba)$ *where* b *is some fixed element and* a *and* c *are arbitrary.*

PROOF. Necessity is obvious. For the sufficiency, let b be given as in the hypothesis and b' be arbitrary. Consider G_b and $G_{b'}$. G_b is commutative by hypothesis. By Theorem 3, Corollary 2, it follows that $G_{b'}$ is also commutative, whence $(ab'c) = (cb'a)$.

Remark. An analogous theorem for sets satisfying Definition 3 does not hold. For if it did then $(ubc) = (cbu) = c(ubu) = cb^{-1}$, whence it would follow that $(abc) = ab^{-1}c$. Yet, in the above example, we have $(xuy) = (yux)$ for all x, y and $(xyz) \neq xy^{-1}z$ in general. As a corollary to this remark we have proved that every commutative system satisfying Definition 3 is a (commutative) abstract coset. We also have complete associativity, that is, $((abc)de) = (a(bcd)e) = (ab(cde))$.[10] Thus any system of this type is not only a Prüfer schar[11] but also a Dörnte 3-group.[12] In fact, the commutative law, A_1 and A_2 (A_3 is then also true) constitute exactly Prüfer's set of postulates. To sum up, we may assert that *in the commutative case the three sets of postulates are equivalent to those of Prüfer and Dörnte and to each other.*

4. Consistency and independence of the sets of postulates. The consistency of these postulates was really proved in the example given above. Actually, any group with $(abc) = ab^{-1}c$ (or $cb^{-1}a$) will satisfy these postulates, with the possible exception of the commutative law. The latter is obviously satisfied if G is commutative.

For the independence, we observe first that the commutative law is obviously independent of the others. To prove the other laws independent we find that a single set of systems will suffice for all three sets of postulates.

Consider the two element systems defined as follows:

$$(abb) = (aaa) = (aba) = (bba) = a, \quad (bbb) = (baa) = (aab) = (bab) = b. \quad (G_1).$$

$$(abb) = (aaa) = (aba) = (aab) = a, \quad (baa) = (bbb) = (bab) = (bba) = b. \quad (G_2).$$

Finally, let G_3 be derived from G_2 by defining $[a'b'c'] = (c'b'a')$.

THEOREM 6. (3.1) *is a set of independent postulates.*

PROOF. Note first that the closure postulate is independent. But G_1 satisfies A_2 and A_3 by inspection. Yet $(ab(bab)) = (abb) = a \neq b$

[10] Pointed out by the referee.

[11] Cf. H. Prüfer, *Theorie der Abelschen Gruppen*, Math. Zeit. vol. 20 (1924) pp. 166–187.

[12] Cf. W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Zeit. vol. 29 (1928) pp. 1–19.

$=(aab)=((abb)ab)$ so that A_1 is independent of A_2 and A_3.

Now consider G_2 and observe that $(xyz)=x$ for all x, y, and z. This gives A_1 and A_3. However A_3 is not satisfied since $(aab)=a\neq b$.

We observe finally that the definition of G_3 from G_2 carries A_1 into itself and permutes A_2 and A_3. For instance $[[a'b'c']\ d'e']$ $=(e'd'(c'b'a'))=((e'd'c')b'a')=[a'b'[c'd'e']]$. This proves A_2 is independent of A_1 and A_3.

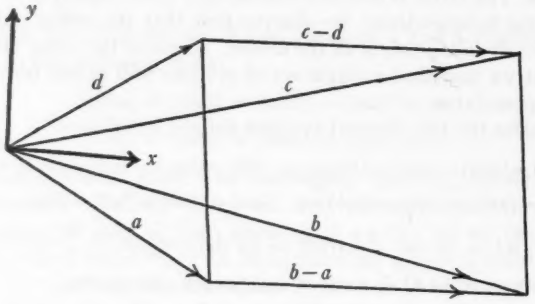
COROLLARY 1. (3.2) *is an independent set of postulates.*

PROOF. We observe first that the systems are symmetric in a and b. The corollary then follows immediately.

COROLLARY 2. (3.3) *is a set of independent postulates.*

PROOF. By symmetry, and proof of Theorem 6, G_1 gives the independence of A. The remainder of the proof follows from the above corollary.

5. **Axioms for free vectors.** By a system of free vectors is usually meant a set of vectors which may be translated without changing their value. For example, the system of forces in physical space constitute such a set. Let us investigate the (commutative) abstract coset derived from the group of all points in the plane under addition, or even a subset which is a coset. We shall then have $(abc)=a-b+c=d$.



By means of the accompanying diagram we see that d is simply the fourth vertex of a parallelogram whose vertices are described cyclicly as a, b, c, and d. For obviously $c-d=b-a$ or $d=a-b+c$. We may thus describe the closure of G as the property of containing with any three points the fourth vertex of the cyclicly ordered parallelogram

(a, b, c, d). This concept generalizes to any group by defining a parallelogram as the ordered quadruple $(a, b, c, ab^{-1}c)$. But we may also consider the difference $b-a$ as a vector (the vector from a to b, say) and interpret the equality $c-d=b-a$ as the statement that parallel vectors of the same length and direction are equal. This is the same as saying that these vectors are free. We shall now define this concept in the abstract.

DEFINITION 5. *By a set of free vectors we mean all pairs of elements (points) of an arbitrary set, the pairs (a, b) being connected by an equivalence relation (reflexive, symmetric and transitive relation) denoted by $\sim$ and satisfying*:

V_1: $(a, b)\sim(a', b')$ *implies* $(b, a)\sim(b', a')$.

V_2: $(a, c)\sim(a', c')$, $(b, c)\sim(b', c')$ *imply* $(a, b)\sim(a', b')$.

V_3: *Given* (a, b) *and* c, *there exists a unique* d *such that* $(a, b)\sim(d, c)$.

These postulates have a very simple interpretation in the light of the preceding paragraph. V_1 simply says that opposite sides of parallelograms are equal when their sense is taken into consideration. V_2 may be interpreted as a statement on congruent triangles and V_3 as a guarantee of the existence of a unique vector through a given point parallel to a given vector.

THEOREM 7. *Given any system of free vectors it is possible to define from them a group.*[13]

Explanation. We shall define as elements of the group the couples ab which are respectively the classes of all pairs equivalent to (a, b). We shall define $ab+cd=ah$, where $(d, c)\sim(h, b)$ and h is given by V_3.

PROOF. We shall divide the proof into several parts.

(a) Given (a, b) and c there exists a unique d such that $(a, b)\sim(c, d)$. By V_3, there exists a d(unique) such that $(b, a)\sim(d, c)$. Apply V_1.

(b) Addition is unique. To prove this we must show that if $(a, b)\sim(a', b')$ and $(c, d)\sim(c', d')$, then $ab+cd=a'b'+c'd'$. But consider h, h' where $(d, c)\sim(h, b)$, $(d', c')\sim(h', b')$. We get by V_1 and hypothesis $(h, b)\sim(h', b')$. V_2 and hypothesis give $(a, h)\sim(a', h')$.

(c) $ab+bc=ac$. By definition, this sum is ah where $(c, b)\sim(h, b)$. The reflexive law and V_3 imply $h=c$, the desired result.

(d) Now let u be fixed. Then uu is a right zero, for by V_3 an arbitrary vector $ab=cu$ for suitable c. Hence $ab+uu=cu+uu=cu=ab$. Again, by (a) $ab=ud$ for suitable d, whence by V_1, $ba=du$. Thus

[13] In connection with this result, which shows that Definition 5 may be considered as a definition of a group, cf. B. A. Bernstein, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 1–6 and H. Boggs and G. Y. Rainich, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 81–84.

$ab+ba=uu$, so ba is a right negative. Since the associative law obviously follows from (c) and (a), the proof is complete.

In the commutative case the postulates may be reduced to a very simple form. We shall prove the following theorem.

THEOREM 8. *Any set of pairs of elements for which there is defined an equivalence relation satisfying the following postulates constitutes a commutative group of free vectors (under the definitions of Theorem 7).*

V_0: $(a, b)\sim(a', b')$ *implies* $(a, a')\sim(b, b')$.

V_2: *Given* (a, b) *and* c *there exists a unique* d *such that* $(a, b)\sim(d, c)$.

PROOF. We first show that these form a system of free vectors. To prove V_1, let $(a, b)\sim(a', b')$. By V_0, we get $(a, a')\sim(b, b')$ or $(b, b')\sim(a, a')$. By V_0 again we have $(b, a)\sim(b', a')$. The hypotheses of V_2 give $(a, a')\sim(b, b')\sim(c, c')$, using V_0. Hence $(a, b)\sim(a', b')$. It remains only to prove that the group which we get is commutative. To this end consider ab and cd and choose e such that $(b, e)\sim(c, d)$ and f such that $(a, b)\sim(e, f)$. But $ab+cd=ae$ while $cd+ab=be+ef=bf$. Since $(a, e)\sim(b, f)$, the result follows.

The question which naturally arises is whether all systems of free vectors (as defined in Definition 5) are necessarily commutative, in which case we should surely use the conditions of Theorem 8. In fact *a necessary and sufficient condition that the group of free vectors be commutative is that* V_0 *hold.* Sufficiency being obvious, suppose $(a, b)\sim(a', b')$. Then $aa'=ab+ba'=ba'+ab=bb'$ since $(a, b)\sim(a', b')$. Hence we get $(a, a')\sim(b, b')$. To settle this question completely and to prove the consistency of these postulates, we now show how any group may be made into a system of free vectors.

THEOREM 9. *Any group* G *with elements* $a, b, \cdots$ *may be converted into a group of free vectors* V, *and conversely. Moreover,* G *and* V *are isomorphic.*

PROOF. *Necessity.* Let G be given and define $(a, b)\sim(c, d)$ if and only if $ab^{-1}=cd^{-1}$. This is obviously an equivalence relation. V_2 is also obvious. But $ab^{-1}=a'b'^{-1}$ implies $ba^{-1}=b'a'^{-1}$ (V_1); and $ac^{-1}=a'c'^{-1}$, $bc^{-1}=b'c'^{-1}$ imply $ab^{-1}=ac^{-1}(bc^{-1})^{-1}=a'c'^{-1}(b'c'^{-1})^{-1}=a'b'^{-1}$, which completes the proof.

Sufficiency. Let V be given and let u be any fixed element. Then the correspondence $a\rightleftarrows au$ is one-one between the elements of the proposed G and the elements of V. We define a binary operation in G by $ab\rightleftarrows au+bu$, thus making G into a group isomorphic to V. Now, by Theorem 7, $a^{-1}\rightleftarrows ua$, and $ab^{-1}\rightleftarrows au+ub=ab$. Thus $ab^{-1}=cd^{-1}$ if and only if $(a, b)\sim(c, d)$.

As for the isomorphism, we need only consider the case where V is defined from G. Obviously, the correspondence $ab^{-1} \rightleftarrows ab$ is one-one from G onto V. Moreover, the correspondent of $ab+cd$ is ah^{-1} where $dc^{-1}=hb^{-1}$ or $h^{-1}=b^{-1}cd^{-1}$. This establishes the isomorphism.

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AN INVARIANT OF INTERSECTION OF TWO SURFACES

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1. **Introduction.** Projective invariants of several pairs of surfaces have been deduced and characterized geometrically by various authors.[1] In this paper we shall supplement their investigations by studying in ordinary space two surfaces intersecting at an ordinary point with distinct tangent planes.

In §2 we show by analysis the existence of a projective invariant determined by the neighborhood of the second order of the two surfaces at the point of intersection.

The final two sections are devoted to the presentation of projectively, as well as metrically, geometric characterizations of this invariant.

2. **Derivation.** Suppose that two surfaces S_1, S_2 in ordinary space intersect at an ordinary point O with distinct tangent planes τ_1, τ_2, and let the common tangent t be distinct from the asymptotic tangents. Let t_1, t_2 be the harmonic conjugate lines of t with respect to the asymptotic tangents of the surfaces S_1, S_2, respectively, at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be, respectively, the axes x, y, z of a general nonhomogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form

$$S_1: \quad y = l_1x^2 + m_1z^2 + \cdots, \tag{1}$$

$$S_2: \quad z = l_2x^2 + m_2y^2 + \cdots. \tag{2}$$

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[1] See the bibliography at the end of the paper.

Let us now consider the most general projective transformation of coordinates which leaves the lines t, t_1, t_2 unchanged:

$$(3)\qquad \begin{cases} x = a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ y = a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \\ z = a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*), \end{cases}$$

where a_{ik} are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$(4)\qquad \begin{cases} a_{33}l_1^* = a_{22}^2 l_1, & a_{33}m_1^* = a_{44}^2 m_1, \\ a_{44}l_2^* = a_{22}^2 l_2, & a_{44}m_2^* = a_{33}^2 m_2, \end{cases}$$

from which it follows immediately that

$$(5)\qquad I = l_1^3 m_2 / l_2^3 m_1$$

is an invariant determined by the neighborhoods of the second order of the two surfaces S_1, S_2 at the point O.

3. **A projectively geometric characterization.** Let us consider two pairs of planes, one of which is determined by the line t_2 and the asymptotic tangents of the surface S_1 at the point O, and the other by the line t_1 and the asymptotic tangents of the surface S_2 at the point O. The lines of intersection of these four planes lie on two planes passing through the line t:

$$(6)\qquad (l_1m_2)^{1/2}y \pm (l_2m_1)^{1/2}z = 0.$$

On the other hand, we observe that the plane sections of the surfaces S_1, S_2 made by any plane through the line t have, in general, an ordinary contact at the point O. In particular, we can uniquely determine a plane through the line t such that the plane sections have contact of the third order at the point O. The equation of this plane is easily found to be

$$(7)\qquad l_2y - l_1z = 0.$$

From equations (6), (7) it follows at once that the double ratio of the four planes τ_1, τ_2, (7), (6) is equal to

$$(8)\qquad (0, \infty, -l_1/l_2, \pm (l_2m_1/l_1m_2)^{1/2}) = \pm I^{1/2}.$$

Hence we obtain the following characterization of the invariant I:

Suppose that two surfaces S_1, S_2 in ordinary space intersect at an ordinary point O with distinct tangent planes τ_1, τ_2, and let the common

tangent t be distinct from the asymptotic tangents. Let t_1, t_2 be the harmonic conjugate lines of t with respect to the asymptotic tangents of the surfaces S_1, S_2, respectively, at the point O; and let π_1 be one of the two planes through t on which lie the lines of intersection of two pairs of planes, one of which is determined by t_2 and the asymptotic tangents of the surface S_1 at the point O, and the other by t_1 and the asymptotic tangents of the surface S_2 at the point O. If π_2 be the unique plane through the line t such that the plane sections of the surfaces S_1, S_2 made by π_2 have contact of the third order at the point O, then the projective invariant I associated with the surfaces S_1, S_2 at the point O is equal to the square of the double ratio $(\tau_1\,\tau_2,\ \pi_2\,\pi_1)$.

4. **A metrically geometric characterization.** It is deemed worthwhile to find a simple metrical characterization of the projective invariant I. For this purpose let us make a projective transformation which leaves the point O unchanged and carries the lines t, t_1, t_2 into three mutually perpendicular lines $\bar{t}$, $\bar{t}_1$, $\bar{t}_2$. Let $\bar{S}_1$, $\bar{S}_2$ be the transformed surfaces of S_1, S_2, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the nonhomogeneous Cartesian coordinates of a point in space referred to the orthogonal coordinate system determined by the lines $\bar{t}$, $\bar{t}_1$, $\bar{t}_2$; then the power series expansions of the surfaces $\bar{S}_1$, $\bar{S}_2$ in the neighborhood of the point O may be written in the form

$$\bar{S}_1\colon\quad \bar{y} = \bar{l}_1\bar{x}^2 + \bar{m}_1\bar{z}^2 + \cdots, \tag{9}$$

$$\bar{S}_2\colon\quad \bar{z} = \bar{l}_2\bar{x}^2 + \bar{m}_2\bar{y}^2 + \cdots, \tag{10}$$

and the projective invariant I takes the form

$$I = \bar{l}_1^3\bar{m}_2/\bar{l}_2^3\bar{m}_1. \tag{11}$$

Let $\bar{K}_1$, $\bar{K}_2$ be the total curvatures of the surfaces $\bar{S}_1$, $\bar{S}_2$ at the point O; and $\bar{R}_1$, $\bar{R}_2$ the radii of curvature at the point O of the curves in which the tangent planes $\bar{z}=0$, $\bar{y}=0$ intersect with the surfaces $\bar{S}_1$, $\bar{S}_2$ respectively. Then it can be demonstrated that

$$\begin{cases} \bar{K}_1 = 4\bar{l}_1\bar{m}_1, & \bar{K}_2 = 4\bar{l}_2\bar{m}_2, \\ \bar{R}_1 = 1/2\bar{l}_1, & \bar{R}_2 = 1/2\bar{l}_2, \end{cases} \tag{12}$$

and therefore that

$$I = \bar{R}_2^4\bar{K}_2/\bar{R}_1^4\bar{K}_1. \tag{13}$$

Thus we obtain the following metrical characterization of the projective invariant I.

Let K_1, K_2 be the total curvatures of the surfaces S_1, S_2 at the point O,

and R_1, R_2 the radii of curvature at O of the curves in which the tangent planes τ_2, τ_1 intersect the surfaces S_1, S_2, respectively. Then $R_2^4K_2/R_1^4K_1$ is the projective invariant I of the surfaces S_1, S_2 above indicated.

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ON THE CONVERGENCE OF CERTAIN PARTIAL SUMS OF A TAYLOR SERIES WITH GAPS

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We consider the function $f(z)$ determined by the power series

$$f(z) = \sum_1^\infty c_n z^{\lambda_n} \tag{1}$$

and its direct analytic continuation. For simplicity, it is supposed that $\limsup |c_n|^{1/\lambda_n} = 1$.

We write

$$S_n(z) = \sum_1^n c_p z^{\lambda_p},$$

$$M(r) = \max_{|z|=r} |f(z)| \qquad (0 < r < 1),$$

$$M(r) = 1 \qquad (r \leqq 0),$$

$$\theta_n = \lambda_{n+1}/\lambda_n - 1.$$

Ostrowski has proved[1] that if $\{\theta_{n_i}\}$ is a sequence extracted from the sequence $\{\theta_n\}$ such that $\liminf \theta_{n_i} > 0$, then every regular point of $f(z)$ on the circle $|z| = 1$ is the center of a circle in which the sequence $\{S_{n_i}(z)\}$ converges uniformly to $f(z)$. Restricting ourselves to the question of convergence at the regular points themselves, we shall prove the following theorem:

If

$$\limsup_{i\to\infty} \frac{\log (M(1-\theta_{n_i}^2)/\theta_{n_i})}{\lambda_{n_i}\theta_{n_i}^2} < \infty, \tag{2}$$

then $\lim S_{n_i}(z) = f(z)$ *at all regular points of* (1) *on the circle* $|z| = 1$.

For the proof, we shall assume that $\lim \theta_{n_i} = 0$; afterwards, we shall remove this restriction, with the aid of Ostrowski's theorem.

Let z_1 be a regular point for (1) on the circle $|z| = 1$, and let z_0 be a point on the segment joining z_1 to the origin. We write $|z_1 - z_0| = a$, and for every positive integer i we define the three circles

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[1] A. Ostrowski, *Über eine Eigenschaft gewisser Potenzreihen mit unendlich vielen verschwindenden Koefficienten*. Preuss. Akad. Wiss. Sitzungsber. vol. 34 (1921) pp. 557–565. Essentially the same proof is to be found in P. Montel's *Leçons sur les familles normales de fonctions analytiques et leurs applications*, pp. 204–207.

$$\Gamma_i' : |z - z_0| = \rho_i' \qquad (0 < \rho_i' < a),$$

$$\Gamma(a) : |z - z_0| = a, \qquad \Gamma_i'' : |z - z_0| = \rho_i'' \qquad (\rho_i'' > a),$$

where ρ_i'' shall be chosen so that the function $f(z)$ is holomorphic in the closed region bounded by Γ_i''. By M_i', $M_i(a)$, and M_i'' we denote the maximum value of $|f(z) - S_{n_i}(z)|$ on Γ_i', $\Gamma(a)$, and Γ_i'', respectively. We write

$$R_i' = 1 + \rho_i' - a, \qquad R_i'' = 1 + \rho_i'' - a,$$

and we choose r_i so that $R_i' < r_i < 1$.

By Cauchy's formula we have, on the circle $|z| = R_i'$,

$$|f(z) - S_{n_i}(z)| \leqq \sum_{p=\lambda_{n_i+1}}^{\infty} \frac{M(r_i)}{r_i^p} R_i'^p = \frac{M(r_i) r_i}{r_i - R_i'} \left(\frac{R_i'}{r_i}\right)^{\lambda_{n_i}(1+\theta_{n_i})},$$

and by the principle of the maximum

$$\log M_i' \leqq \log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i}.$$

On the circle $|z| = R_i''$, and therefore also on Γ_i'', we have

$$|S_{n_i}(z)| \leqq M(r_i) \sum_0^{\lambda_{n_i}} \left(\frac{R_i''}{r_i}\right)^p < \frac{M(r_i) R_i''}{R_i'' - r_i} \left(\frac{R_i''}{r_i}\right)^{\lambda_{n_i}}.$$

Since $f(z)$ is holomorphic in the closed region bounded by Γ_i'', and since at least one of the expressions $R_i''/(R_i'' - r_i)$, $(R_i''/r_i)^{\lambda_{n_i}}$ tends to ∞ as $i \to \infty$, we may write, for any positive η and for i sufficiently large,

$$\log M_i'' < \log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i}.$$

Applying Hadamard's three-circle theorem to the function $S_{n_i}(z) - f(z)$ on the circles Γ_i', $\Gamma(a)$, Γ_i'', we have now

$$\log \frac{\rho_i''}{\rho_i'} \log M_i(a) \leqq \log \left(\frac{\rho_i''}{a}\right) \log M_i' + \log \left(\frac{a}{\rho_i'}\right) \log M_i''$$

$$< \log \left(\frac{\rho_i''}{a}\right) \left\{ \log \frac{M(r_i) r_i}{r_i - R_i'} + \lambda_{n_i}(1 + \theta_{n_i}) \log \frac{R_i'}{r_i} \right\} \tag{3}$$

$$+ \log \left(\frac{a}{\rho_i'}\right) \left\{ \log \frac{M(r_i)(1 + \eta) R_i''}{R_i'' - r_i} + \lambda_{n_i} \log \frac{R_i''}{r_i} \right\}.$$

In the proof of his theorem Ostrowski now takes the point z_0 near

to the point z_1 and chooses for Γ_i', $\Gamma(a)$, and Γ_i'' three fixed circles with radii sufficiently near to $|z_1-z_0|$ (with $\Gamma(a)$ including the point z_1 instead of passing through it). In our case it is necessary to take z_0 near to the origin and to let ρ_i' and ρ_i'' tend to a as i becomes large. We choose $\rho_i'=(1-b_i)a$, $\rho_i''=(1+b_i)a$, $r_i=1-k_i$, where $0<b_i<1$ and $0<k_i<ab_i$; substituting these values in (3), expanding terms such as $\log(1+ab_i)$ in power series, and dividing both sides of the inequality by b_i, we get

$$
\begin{aligned}
&2(1+b_i^2/3+\cdots)\log M_i(a) < (1-b_i/2+b_i^2/3-\cdots)\\
&\quad\cdot\left\{\log\frac{M(1-k_i)(1-k_i)}{(1-k_i/ab_i)ab_i}+\lambda_{n_i}(1+\theta_{n_i})\log\frac{1-ab_i}{1-k_i}\right\}\\
&\quad+(1+b_i/2+b_i^2/3+\cdots)\\
&\quad\cdot\left\{\log\frac{M(1-k_i)(1-k_i)(1+\eta)}{(1+k_i/ab_i)ab_i}+\lambda_{n_i}\log\frac{1+ab_i}{1-k_i}\right\}\\
&<2\eta+3k_i+3k_i/ab_i-3\log a\\
&\quad+2(1+b_i^2/3+\cdots)\log\frac{M(1-k_i)}{b_i}\\
&\quad+\lambda_{n_i}\{\theta_{n_i}[1-b_i/2+b_i^2/3-\cdots]\\
&\qquad\cdot[-ab_i-a^2b_i^2/2-\cdots+k_i+k_i^2/2+\cdots]\\
&\quad+[1-b_i/2+b_i^2/3-\cdots][-ab_i-a^2b_i^2/2-\cdots]\\
&\quad+[1+b_i/2+b_i^2/3+\cdots][ab_i-a^2b_i^2/2+\cdots]\\
&\quad+2[1+b_i^2/3+\cdots][k_i+k_i^2/2+\cdots]\}\\
&<2\eta+3k_i+3k_i/ab_i-3\log a+3\log\frac{M(1-k_i)}{b_i}\\
&\quad-(\lambda_{n_i}/2)\{ab_i[\theta_{n_i}-b_i(1-a)]-3k_i\}
\end{aligned}
\tag{4}
$$

provided θ_{n_i} and b_i are sufficiently small and the sum of the terms in the braces of the last term is positive.

Now suppose that

$$
\limsup_{i\to\infty}\frac{\log(M(1-\theta_{n_i}^2)/\theta_{n_i})}{\lambda_{n_i}\theta_{n_i}^2}<H\qquad(1<H<\infty).
$$

Choose $b_i=b\theta_{n_i}$, $k_i=b_i\theta_{n_i}/b=\theta_{n_i}^2$, where $b>14H$. The last member of (4) becomes

$$
-(\lambda_{n_i}\theta_{n_i}^2 b/2)\{a[1-b(1-a)]-3/b\}.
$$

If a is chosen sufficiently near to 1, $b\{a[1-b(1-a)]-3/b\}>10H$, and for sufficiently large values of i (4) becomes

$$(2+\epsilon)\log M_i(a) < K(a,b) + 3\log M(1-\theta_{n_i}^2)/\theta_{n_i} - 5\lambda_{n_i}\theta_{n_i}^2 H,$$

where $K(a, b)$ depends on a and b only. But $\lim_{i\to\infty}\theta_{n_i}=0$ implies, together with the validity of (2), that $\lim_{i\to\infty}\lambda_{n_i}\theta_{n_i}^2=\infty$. For sufficiently large i we have

$$(2+\epsilon)\log M_i(a) < -\lambda_{n_i}\theta_{n_i}^2 H,$$

that is,

$$\lim_{i\to\infty}\log M_i(a) = -\infty,$$

and, in particular,

$$\lim_{i\to\infty} S_{n_i}(z_1) = f(z_1).$$

Now let $\{\theta_{n_i}\}$ be any sequence of values θ_{n_i} for which (2) is satisfied, and let z_1 be a regular point for (1). From every subsequence of $\{\theta_{n_i}\}$ we can extract a further subsequence $\{\theta_{m_j}\}$ such that either $\lim \theta_{m_j}=0$ or $\liminf \theta_{m_j}>0$. In the first case, $\lim S_{m_j}(z_1)=f(z_1)$ by what we have just proved; in the second case, by Ostrowski's theorem. From every subsequence of $\{n_i\}$ we can therefore extract a further subsequence $\{m_j\}$ such that $\lim S_{m_j}(z_1)=f(z_1)$. It follows that the sequence $\{S_{n_i}(z_1)\}$ itself tends to $f(z_1)$, and our theorem is proved.

Condition (2) may be replaced by one that is somewhat less general, but can be expressed more immediately in terms of the behavior of $M(r)$:

If $\liminf \theta_{n_i}=\theta$, we define, for $h>\theta$,

$$\lambda(h) = \text{g.l.b.}_{\theta_{n_i}\leqq h}\ \lambda_{n_i};$$

for $0\leqq h\leqq\theta$, we write $\lambda(h)=\infty$. It follows from our theorem that $\lim S_{n_i}(z_1)=f(z_1)$ whenever z_1 is a regular point for (1) on the circle $|z|=1$, provided

$$\limsup_{h\to 0+}\frac{\log\left(M(1-h^2)/h\right)}{h^2\lambda(h)} < \infty. \tag{5}$$

If we write $r=1-h^2$, (5) becomes

$$\limsup_{r\to 1-0}\frac{\log\left[M(r)/(1-r)^{1/2}\right]}{(1-r)\lambda(1-r)^{1/2}} < \infty.$$

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THE RICE INSTITUTE

ON ABEL AND LEBESGUE SUMMABILITY

OTTO SZÁSZ

1. **Introduction.** A series $\sum_1^\infty a_n$ is called Abel summable to the value s if the power series $\sum a_n r^n$ converges for $0<r<1$, and if $\sum a_n r^n \to s$ as $r \uparrow 1$; it is called Lebesgue summable if the sine series

$$\sum_1^\infty a_n \frac{\sin nt}{n} = F(t) \tag{1.1}$$

converges in some interval $0<t<\tau$, and if

$$t^{-1}F(t) \to s \quad \text{as} \quad t \downarrow 0. \tag{1.2}$$

We write in the first case $A\sum a_n = s$, and in the latter case $L\sum a_n = s$ (summability A or L respectively). It is known that convergence does not imply L-summability and conversely L-summability does not imply convergence of $\sum a_n$. Tauberian type problems which arise out of this situation have been discussed.[1] It is also known that either convergence or L-summability imply A-summability. As to the converse (restricting ourselves to real a_n) we have proved the following theorems:

THEOREM 1. [8, pp. 582–583]. *If*

$$\sum_n^{2n} (|a_\nu| - a_\nu) = O(1) \quad as \quad n \to \infty, \tag{1.3}$$

and if

$$\sum_1^\infty a_n r^n = O(1) \quad as \quad r \uparrow 1, \tag{1.4}$$

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[1] See [8], where further references are given; numbers in brackets refer to the bibliography at the end of this paper.

then

$$t^{-1}F(t) = O(1) \quad as \quad t \downarrow 0. \tag{1.5}$$

THEOREM 2. [8, p. 585]. *If* (1.3) *holds and if*

$$\lim_{\lambda \downarrow 1} \liminf_{n \to \infty} \min_{n \le k \le \lambda n} \sum_{n}^{k} a_\nu \geqq 0, \tag{1.6}$$

then A-summability implies L-summability.

Note that A-summability and (1.6) (without (1.3), which need not be satisfied) imply convergence (by a theorem of R. Schmidt) and are also necessary for convergence, while the series need not be L-summable.

We remark also that, in the assumption and in the conclusion of Theorem 1, $O(1)$ can be replaced by $o(1)$; for if

$$\sum_{n}^{2n} (|a_\nu| - a_\nu) = o(1) \quad \text{as} \quad n \to \infty, \tag{1.7}$$

then (1.6) holds. Moreover by the previous remark the series $\sum a_n$ converges (to zero).

We shall complete and generalize these results by proving the following theorems:

THEOREM 3. *If* (1.3) *holds then each of the statements* (1.4), (1.5) *and*

$$\sum_{1}^{n} a_\nu = O(1) \quad as \quad n \to \infty \tag{1.8}$$

implies the two others.

THEOREM 4. *If* (1.3) *holds then A-summability implies L-summability, but not necessarily convergence.*

THEOREM 5. *If* (1.3) *holds and if* $\sum a_n$ *converges, then* $\sum a_n \sin nt/nt$ *converges uniformly in* $0 < t < \pi$.

This generalizes Theorem 6′ of my paper [8].

2. **Proof of Theorem 3.** We prove the following lemma.

Lemma 1. *If* (1.3) *and* (1.4) *hold, then*

$$s_n = \sum_{1}^{n} a_\nu = O(1), \qquad \sum_{n}^{2n} |a_\nu| = O(1), \qquad \sum_{1}^{n} \nu |a_\nu| = O(n), \tag{2.1}$$

$$\sum_{1}^{\infty} \nu^{-1} |a_\nu| < \infty, \qquad \sum_{n}^{\infty} \nu^{-1} |a_\nu| = O(n^{-1}) \quad as \quad n \to \infty. \tag{2.2}$$

The statement $s_n=O(1)$ is an immediate corollary of a previous result [6, Lemma 2]. Combining it with (1.3) we get

$$\sum_{n}^{2n}|a_\nu| = \sum_{n}^{2n}(|a_\nu| - a_\nu) + s_{2n} - s_{n-1} = O(1) \quad \text{as} \quad n\to\infty.$$

Furthermore, where $\sum_\alpha^\beta$ means summation over the range $\alpha<\nu\leqq\beta$,

$$\sum_{1}^{n}\nu|a_\nu| = \sum_{k=0}^{n}\sum_{n/2^{k+1}}^{n/2^k}\nu|a_\nu| \leqq \sum_{k=0}^{n}\left(\frac{n}{2^k}\sum_{n/2^{k+1}}^{n/2^k}|a_\nu|\right)$$

$$= O\left(n\sum_{0}^{\infty}2^{-k}\right) = O(n).$$

(2.1) is now proved. We have thus $\sum_n^{2n}|a_\nu|<c$, a positive constant, and $\sum_n^{2n}\nu^{-1}|a_\nu|<c/n$, hence

$$\sum_{1}^{n}\nu^{-1}|a_\nu| \leqq \sum_{k=1}^{n}\sum_{2^{k-1}}^{2^k}\nu^{-1}|a_\nu| < c\sum_{k=1}^{\infty}2^{1-k} = 2c.$$

This proves the first part of (2.2). Finally

$$\sum_{n}^{\infty}\nu^{-1}|a_\nu| \leqq \sum_{k=1}^{\infty}\sum_{n\cdot 2^{k-1}}^{n\cdot 2^k}\nu^{-1}|a_\nu| < \frac{c}{n}\sum_{1}^{\infty}2^{1-k} = \frac{2c}{n},$$

which proves the lemma.

We now prove Theorem 3. If (1.3) holds, then (1.8) implies (1.5) by Theorem 5 of my paper [8], and (1.4) follows from the remark to the same theorem. By the same remark (1.4) implies (1.8), hence also (1.5). Finally, assuming (1.5), to prove (1.8) we write

$$t^{-1}F(t) - s_n = \sum_{1}^{n}a_\nu\left(\frac{\sin\nu t}{\nu t} - 1\right) + \sum_{n+1}^{\infty}a_\nu\frac{\sin\nu t}{\nu t} \equiv S_1 + S_2.$$

From $0<1-\sin\nu t/\nu t<\nu^2t^2$ we get

$$|S_1| < t^2\sum_{1}^{n}\nu^2|a_\nu| < nt^2\sum_{1}^{n}\nu|a_\nu| = t^2O(n^2);$$

furthermore, by Lemma 1,

$$|S_2| < t^{-1}\sum_{n}^{\infty}\nu^{-1}|a_\nu| = O(n^{-1}t^{-1}).$$

On putting now $t=n^{-1}$ we get

$$nF(n^{-1}) - s_n = O(1) \quad \text{as} \quad n\to\infty;$$

this proves (1.8) and a fortiori (1.4), which completes the proof of Theorem 3.

3. **Proof of Theorem 4.** We first prove the following lemmas.

LEMMA 2. *Let*

$$\Delta_n = \sin nt/nt - \sin (n+1)t/(n+1)t,$$
$$\Delta_n^2 = \Delta(\Delta_n) = \sin nt/nt - 2\sin (n+1)t/(n+1)t + \sin (n+2)t/(n+2)t;$$

then

$$0 < \Delta_n^2 < t^2 \quad \textit{for} \quad (n+2)t < \pi/2, \tag{3.1}$$

$$|\Delta_n| < 2/n \quad \textit{for} \quad nt > 1. \tag{3.2}$$

Applying the mean value theorem to Δ^2 we get easily (see [8, Lemma 4])

$$0 < \Delta_n^2 < t^2 \quad \text{for} \quad (n+2)t < \pi/2.$$

Furthermore

$$\Delta_n = \frac{\sin (n+1)t}{n(n+1)t} - 2\frac{\sin (t/2)\cos ((2n+1)t/2)}{nt},$$

which yields

$$|\Delta_n| < 1/n(n+1)t + 1/n < 2/n \qquad \text{for } nt > 1.$$

LEMMA 3. *If $\sum a_n$ is Abel summable and if* (1.3) *holds, then $\sum a_n$ is Cesàro summable of any order $\alpha > 0$.*

By Lemma 1, $s_n = O(1)$; this and A-summability imply $(C, 1)$ summability, as was proved first by Littlewood in 1910. For a short proof (with a more general assumption) cf. [5]. That Abel summability and $s_n = O(1)$ imply (C, α) summability for any $\alpha > 0$ has been proved by Andersen [1, p. 80]. We shall apply only the case $\alpha = 1$.

Let now $\sum_1^n s_\nu = s_n'$, then $n^{-1}s_n'$ tends to a limit s; we can assume without loss of generality that $s = 0$ (otherwise replace a_1 by $a_1 - s$). To a given positive $\epsilon < 1/2$ we now choose $n_0(\epsilon)$ so that

$$|s_n'| < \epsilon^3 n \quad \text{for} \quad n > n_0(\epsilon) > 3. \tag{3.3}$$

By (2.2) $\sum \nu^{-1}a_\nu \sin \nu t$ converges absolutely; we write

$$t^{-1}F(t) = \sum_1^\infty a_\nu \frac{\sin \nu t}{\nu t} = \sum_1^n + \sum_{n+1}^\infty \equiv T_1 + T_2.$$

We restrict ourselves to $0<t<n_0^{-1}$, and choose $n=1+[\epsilon^{-1}t^{-1}] > \epsilon^{-1}t^{-1}>\epsilon^{-1}n_0>2n_0$; Abel's summation by parts yields

$$T_1 = s_n \frac{\sin nt}{nt} + s'_{n-1}\Delta_{n-1} + \sum_1^{n-2} s'_\nu \Delta^2_\nu.$$

Now $nt>\epsilon^{-1}$. Hence

$$|s_n \sin nt/nt| < |s_n|/nt < \epsilon|s_n| = \epsilon O(1) \quad \text{as} \quad t\downarrow 0, \tag{3.4}$$

and, from (3.2) and (3.3),

$$|s'_{n-1}\Delta_{n-1}| < 2\epsilon^3; \tag{3.5}$$

furthermore

$$|T_2| < t^{-1}\sum_n^\infty \nu^{-1}|a_\nu| = O(n^{-1}t^{-1}) = O(\epsilon) \quad \text{as} \quad t\downarrow 0. \tag{3.6}$$

Finally, write

$$\sum_1^{n-2} s'_\nu \Delta^2_\nu = \left(\sum_1^{k-1} + \sum_k^{n-2}\right) s'_\nu \Delta^2_\nu, \qquad 2 \leqq k \leqq n-2,$$

and choose

$$k = 1 + [t^{-1}] > t^{-1} > n_0(\epsilon) > 3.$$

By (3.1), as $(k+1)t<(2+t^{-1})t<3/2<\pi/2$,

$$\left|\sum_1^{k-1} s'_\nu \Delta^2_\nu\right| < t^2 \sum_1^k |s'_\nu| = o(t^2k^2) = o(1). \tag{3.7}$$

It remains to estimate $\sum_k^{n-2} s'_\nu \Delta^2_\nu$. We decompose this sum according to the changes of sign of the factors Δ^2_ν, and write

$$\sum_k^{n-2} s'_\nu \Delta^2_\nu = \sum_1 + \sum_2 + \cdots + \sum_\rho.$$

To estimate ρ we note that there are not more changes of sign in the sequence Δ^2_ν than there are zeros $x_1, x_2, \cdots$ of $D_2(x^{-1}\sin x)$ in the interval $0<x<(n-1)t$. A simple calculation yields for x_ν the estimate

$$x_\nu = (\nu+1)\pi - \psi_\nu, \quad 0<\psi_\nu<\pi/3, \quad \nu = 1, 2, 3, \cdots;$$

hence,

$$\rho\pi < x_\rho < (n-1)t < \epsilon^{-1}.$$

But each $\sum$ is in absolute value less than $4\epsilon^3 nk^{-1}$ (from (3.2) and (3.3)), and

$$\epsilon^3 nk^{-1} < \epsilon^3 nt < 2\epsilon^2;$$

thus

$$\left| \sum_{k}^{n-2} s_\nu' \Delta_\nu^2 \right| < 2\rho\epsilon^2 < \epsilon. \tag{3.8}$$

Collecting the estimates (3.4) to (3.8) we find

$$|t^{-1}F(t)| < \epsilon O(1) + o(1) \quad \text{as} \quad t \downarrow 0;$$

ϵ being arbitrarily small the positive part of Theorem 4 follows. For the negative part we refer to the examples in §5.

4. **Proof of Theorem 5.** We write, for $\lambda > 1$,

$$\sum_{n+1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t} = \sum_{n+1}^{\lambda n} + \sum_{\nu > \lambda n} = R_1 + R_2,$$

say; then by (2.2)

$$|R_2| < t^{-1} \sum_{\nu > \lambda n} \nu^{-1} |a_\nu| = \frac{1}{\lambda nt} O(1).$$

Abel's summation by parts yields

$$\sum_{1}^{n} a_\nu \frac{\sin \nu t}{\nu t} = s_n \frac{\sin nt}{nt} + \sum_{1}^{n-1} s_\nu \Delta_\nu,$$

whence

$$\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t} = s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} + \sum_{n}^{n+k-1} s_\nu \Delta_\nu.$$

We may assume that the limit of s_n is zero; given $\epsilon > 0$, we choose $n_0(\epsilon)$ so that $|s_n| < \epsilon^3$ for $n > n_0$; then

$$\left| s_{n+k} \frac{\sin (n+k)t}{(n+k)t} - s_n \frac{\sin nt}{nt} \right| < 2\epsilon^3 \quad \text{for} \quad n > n_0(\epsilon).$$

We define k by $n+k = [\lambda n]$, thus $k = [\lambda n] - n \leqq (\lambda - 1)n$. We subdivide the range $n \leqq \nu < \lambda n$ into consecutive parts in each of which Δ_ν has constant sign; denote the number of subdivisions by σ. Denoting the positive zeros of $u^{-1} \sin u$ by $u_1 < u_2 < \cdots$, we find easily $u_\nu = \nu\pi + \alpha_\nu$, where $0 < \alpha_\nu < \pi/2$; the number of zeros in the interval $nt < u < \lambda nt$ is therefore less than $2\lambda nt/\pi$, and

$$\sigma \leqq \lambda nt + 2.$$

In each section $|\sum s_\nu\Delta_\nu| < 2\epsilon^3$, hence

$$\left|\sum_{n}^{n+k-1} s_\nu\Delta_\nu\right| < 2\epsilon^3(2+\lambda nt),$$

and

$$|R_1| < 2\epsilon^3(3+\lambda nt).$$

We now choose $\lambda = 1/\epsilon^2 nt$, for whatever $n > n_0(\epsilon)$ and any $0 < t < \pi$, if $\epsilon^2 nt < 1$, and put $\lambda = 1$ (that is $R_1 \equiv 0$) otherwise. In the latter case $|\sum_{n+1}^{\infty} a_\nu \sin(\nu t)/\nu t| < (nt)^{-1}O(1) < \epsilon^2 O(1)$, while in the first case

$$\left|\sum_{n+1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon^2 O(1) + 2\epsilon^3\left(3+\frac{1}{\epsilon^2}\right) < \epsilon O(1)$$

for $n > n_0(\epsilon)$ and $0 < t < \pi$. This proves our theorem.

Note that convergence of $\sum a_n$ is a necessary condition for the uniform convergence of $\sum a_n \sin(nt)/nt$. For if, for any $\epsilon > 0$,

$$\left|\sum_{n+1}^{n+k} a_\nu \frac{\sin \nu t}{\nu t}\right| < \epsilon \quad \text{for} \quad n > n_0(\epsilon), \quad k = 1, 2, 3, \cdots, 0 < t < \pi,$$

then, letting $t \downarrow 0$ we get $|\sum_{n+1}^{n+k} a_\nu| \leq \epsilon$. Moreover we have uniform convergence in the closed interval.

It is shown easily that the assumption (1.3) is equivalent to either of the following conditions: There exists a constant $\lambda > 1$ such that

$$\sum_{n}^{\lambda n} (|a_\nu| - a_\nu) = O(1); \tag{4.1}$$

$$\sum_{1}^{n} \nu(|a_\nu| - a_\nu) = O(n), \quad \text{as} \quad n \to \infty. \tag{4.2}$$

For a more general statement see [7, p. 129].

A consequence of our results is the following theorem:

THEOREM 6. *If*

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{4.3}$$

then A-summability of $\sum a_n$ *implies uniform convergence of the series* $\sum a_n \sin(nt)/nt$ *in* $0 < t < \pi$.

Clearly (4.3) implies (4.1), whence (1.3). Now, by Theorem 4, $\sum a_n$ is L-summable; furthermore by Theorem 4 of our paper [8] L-summability and (4.3) imply convergence of $\sum a_n$. Theorem 6 now follows from Theorem 5.

5. **Negative results.** We quote the following lemma.

LEMMA 4. *Let* $n \geqq 1$ *and*

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \cdots - \frac{z^{2n-1}}{n};$$

then, when $|z| \leqq 1$,

$$|P_n(z)| < 6.$$

For the proof see Fejér [2, pp. 36–37].

Consider the polynomial series $\sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$, where $\lambda_1 = 1$, $k_1 = 3$, $2\lambda_n = 2^{n^2}$, $2k_n = \lambda_{n+1} - \lambda_n$, $n \geqq 2$. In view of the above lemma the series converges uniformly in $|z| \leqq 1$, so that the function

$$F(z) = \sum_1^\infty n^{-2} z^{\lambda_n} P_{k_n}(z)$$

is regular in $|z| < 1$ and continuous in $|z| \leqq 1$. The degree of the nth term is $2k_n + \lambda_n - 1 < \lambda_{n+1}$, hence writing out the polynomials explicitly we get a power series, convergent for $|z| < 1$,

$$F(z) = \sum a_n z^n. \tag{5.1}$$

For $|z| = 1$ we get a Fourier power series of a *continuous* function $F(e^{it})$. The structure of P_n and the inequality $(n+1)^{-2} \log k_n < \log 2$ easily yield

$$\sum_n^{2n} |a_\nu| = O(1) \quad as \quad n \to \infty.$$

But $\sum a_n$ diverges, as there are sections $\sum a_\nu = n^{-2} \sum_1^{k_n} 1/\nu$ which do not tend to zero. On the other hand the series (5.1) is evidently L-summable at every point on $|z| = 1$.

Next we define a series $\sum a_n$ by putting $s_n = 1$ for $n = 2^k$, $k = 0, 1, 2, \cdots$, and $s_n = 0$ otherwise. Now $n^{-1} \sum_1^n s_\nu \to 0$, moreover $\sum_n^{2n} |a_\nu| \leqq 3$, hence the series is summable L. But $\sum a_n$ diverges, in fact lim sup $|a_n| = 1$, and $\sum a_n \cos nt$ is not a Fourier series.

Another example of this kind is due to Neder [4].

In contrast Menchoff [3] tried to prove that A-summability and (1.3) imply convergence of $\sum a_n$; the error lies in his Lemma 4 which is false. It is based on a false interpretation of an argument used by Landau.

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ON SUBHARMONIC FUNCTIONS

MAXWELL READE

If $p(x, y)$ is continuous in a domain (non-null connected open set) $\mathcal{D}$, then $p(x, y)$ is subharmonic in $\mathcal{D}$ if and only if the inequality

$$p(x, y) \leqq A(p; x, y; r) \equiv \frac{1}{\pi r^2}\int\int_{D(x,y;r)} p(x+\xi, y+\eta)d\xi d\eta$$

holds for all circular discs $D(x_0, y_0; r): (x-x_0)^2+(y-y_0)^2=\xi^2+\eta^2\leqq r^2$ in $\mathcal{D}$. If $p(x, y)$ is continuous along with its partial derivatives of the second order in $\mathcal{D}$, then $p(x, y)$ is subharmonic there if and only if $\Delta p(x, y)\geqq 0$ in $\mathcal{D}$, where Δ is the Laplace operator [2, 3].[1]

If $p(x, y)$ is continuous in $\mathcal{D}$, then $p(x, y)$ is said to be of class PL [2] in $\mathcal{D}$ provided (i) $p(x, y)\geqq 0$ and (ii) $\log p(x, y)$ is subharmonic wherever $p(x, y)\neq 0$. If $p(x, y)\geqq 0$ and is continuous along with its partial derivatives of the second order, then $p(x, y)$ is of class PL if and only if $p\Delta p-p_x^2-p_y^2\geqq 0$ wherever $p(x, y)\neq 0$.

Beckenbach [1] has proved the following theorem characterizing functions of class PL.

THEOREM A. *If $p(x, y)\geqq 0$ in $\mathcal{D}$, then $p(x, y)$ is of class PL in $\mathcal{D}$ if and only if $[(x-\alpha)^2+(y-\beta)^2]p(x, y)$ is subharmonic in $\mathcal{D}$ for every choice of the real constants α, β.*

The Beckenbach theorem is comparable to the classic Montel-Radó theorem, which was later generalized by Kierst and Saks [3, 4] for functions $p(x, y)$ with continuous partial derivatives of the second order; this generalization is the following theorem.

THEOREM B. *Let $f(t)$ have a continuous second derivative, with $f'(t)>0$, for $-\infty<t<\infty$. If $v(x, y)$ has continuous partial derivatives of the second order in $\mathcal{D}$, and if the function $f(\alpha x+\beta y+v(x, y))$ is subharmonic in $\mathcal{D}$ for every choice of the real constants α, β, then $v(x, y)$ is subharmonic in $\mathcal{D}$.*

The question arises as to the possibility of exhibiting a Kierst-Saks type of generalization for Beckenbach's Theorem A. Our result is the following.

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[1] The numbers in square brackets refer to references listed in bibliography at end of paper.

THEOREM C. *Let $f(t)$ have a continuous second derivative, with $f'(t)>0$, for $-\infty<t<\infty$. If $v(x, y)$ has continuous partial derivatives of the second order in $\mathcal{D}$, and if the function*

$$\Phi(v; \alpha, \beta; x, y) \equiv f\{\log [(x-\alpha)^2+(y-\beta)^2]+v(x, y)\} \tag{1}$$

is subharmonic in $\mathcal{D}$ for every choice of the real constants α, β, then $v(x, y)$ is subharmonic in $\mathcal{D}$.

PROOF. We shall show $\Delta v(x, y)\geqq 0$. From the hypothesis, it follows that $\Phi(v; \alpha, \beta; x, y)$ has a non-negative Laplacian, that is,

$$f'(t)\Delta v(x, y)+f''(t)\left\{\left[\frac{2(x-\alpha)}{(x-\alpha)^2+(y-\beta)^2}+v_x(x, y)\right]^2+\left[\frac{2(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}+v_y(x, y)\right]^2\right\}\geqq 0, \tag{2}$$

where $t\equiv\log [(x-\alpha)^2+(y-\beta)^2]+v(x, y)$, for all real α, β.

Let (x_0, y_0) be a point of $\mathcal{D}$. We distinguish two cases. (a) $v_x^2+v_y^2\neq 0$ at (x_0, y_0). Then it is sufficient to choose

$$\alpha = x_0+2v_x/(v_x^2+v_y^2), \qquad \beta = y_0+2v_y/(v_x^2+v_y^2),$$

where the partial derivatives are evaluated at (x_0, y_0), and then to make use of the positiveness of $f'(t)$ to obtain $\Delta v(x_0, y_0)\geqq 0$ from (2). (b) $v_x^2+v_y^2=0$ at (x_0, y_0). If (x_0, y_0) can be approached by a sequence of points (x_n, y_n), $n=1, 2, \cdots$, at each of which $v_x^2+v_y^2\neq 0$, then it follows from (a) and the continuity of $\Delta v(x, y)$ that $\Delta v(x_n, y_n)\to\Delta v(x_0, y_0)\geqq 0$, as $n\to\infty$. If no such sequence exists, then $v_x^2+v_y^2\equiv 0$ in a neighborhood of (x_0, y_0), so that $\Delta v(x_0, y_0)=0$. This completes the proof.

Beckenbach has also given an isoperimetric characterization of functions of class PL [2]; if we use the abbreviation

$$L(p; x, y; r) \equiv \frac{1}{2\pi r}\int_{C(x, y; r)} p(x+\xi, y+\eta)ds$$

where $C(x, y; r)$ is the boundary of $D(x, y; r)$, then the theorem is the following.

THEOREM D. *If $p(x, y)$ is continuous and if $p(x, y)\geqq 0$ in $\mathcal{D}$, then $p(x, y)$ is of class PL in $\mathcal{D}$ if and only if, for every continuous function $q(x, y)$ of class PL in $\mathcal{D}$, the inequality*

$$A(pq; x, y; r)\leqq L(p; x, y; r)\cdot L(q; x, y; r),$$

holds for every circular disc $D(x, y; r)$ in $\mathcal{D}$.

This latter theorem may be extended as follows, for functions that have continuous second derivatives in $\mathcal{D}$.

THEOREM E. *If $p(x, y)$ has continuous partial derivatives of the second order in $\mathcal{D}$, if $p(x, y) \geqq 0$ in $\mathcal{D}$, and if a, b is a pair of real constants such that $(1/a)+(1/b)=2$, then $p(x, y)$ is of class PL in $\mathcal{D}$ if and only if, for every continuous function $q(x, y)$ of class PL in $\mathcal{D}$, the inequality*

$$A(pq;\, x,\, y;\, r) \leqq [L(p^a;\, x,\, y,\, r)]^{1/a} \cdot [L(q^b;\, x,\, y;\, r)]^{1/b} \tag{3}$$

holds for every circular disc $D(x, y; r)$ in $\mathcal{D}$.

Necessity. A proof of the necessity part may be given that parallels Beckenbach's proof of his theorem; indeed, the only change would be to use the more general Hölder inequality where Beckenbach uses the Schwarz inequality. It is to be noted that the hypothesis on the existence of any derivative of $p(x, y)$ is superfluous here.

Sufficiency. Let (x_0, y_0) be a point of $\mathcal{D}$ such that $p(x_0, y_0) \neq 0$; if there were no such point, then $p(x, y) \equiv 0$ and hence of class PL in $\mathcal{D}$. We shall consider only analytic $q(x, y)$, so that we may expand both $p(x, y)$ and $q(x, y)$ in finite Taylor expansions about (x_0, y_0), and then substitute in (3) to obtain

$$pq + (r^2/8)\Delta(pq)$$
$$\leqq [p^a + (r^2/4)\Delta(p^a) + o(r^2)]^{1/a}[q^b + (r^2/4)\Delta(q^b) + o(r^2)]^{1/b} + o(r^2),$$

where $o(r^2)$ is a quantity (not always the same quantity) such that $[o(r^2)/r^2] \to 0$, as $r \to 0$. We use binomial expansions, for small r, and *then* let $r \to 0$, to obtain the further inequality

$$\Delta(pq)/8 \leqq (pq/4ap^a)\Delta(p^a) + (pq/4bq^b)\Delta(q^b), \tag{4}$$

where the derivatives are evaluated at (x_0, y_0). If we express the derivatives of pq, p^a and q^b in terms of the derivatives of p and q, and if we set $q \equiv e^{\alpha x + \beta y}$, we obtain

$$\begin{aligned} 0 \leqq\ & (2b-1)(\alpha^2+\beta^2)p^2 - 2\alpha p p_x \\ & - 2\beta p p_y + p\Delta p + (2a-2)(p_x^2 + p_y^2), \end{aligned} \tag{5}$$

which must hold for all real constants α, β. Hence the discriminant of the right-hand member of (5) must satisfy

$$0 \leqq (2b-1)^2[p\Delta p + (2a-2)(p_x^2 + p_y^2)] - (2b-1)(p_x^2 + p_y^2),$$

which yields, since $2b-1>0$ and $(1/a)+(1/b)=2$,

$$0 \leqq p\Delta p - p_x^2 - p_y^2.$$

Hence $p(x, y)$ is of class PL in $\mathcal{D}$.

The usual averaging process does not appear to be adaptable to the lessening of differentiability conditions in Theorems B, C and E.

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A NOTE ON SUBSEQUENCES

R. CREIGHTON BUCK

It is well known that a sequence is convergent if all of its subsequences are convergent. The purpose of this note is to show that this may be extended to a result of great generality.

THEOREM. *A sequence is convergent if there exists a regular matrix summability method T, which sums every subsequence of the given sequence.*

Thus, for example, if every subsequence of a given sequence is Cesàro summable of any order, then the sequence is convergent. The general theorem depends upon the following well known result of H. Steinhaus.[1]

LEMMA. *For any regular summability method T, given by the matrix (A_{nm}), there exists a sequence of 0's and 1's not summable T.*

To establish the theorem, we suppose that the given sequence $\{S_n\}$ is divergent; we then show that, for any T, we can choose a subsequence $\{S_{r_n}\}$ not summable T. We may obviously restrict ourselves to bounded sequences. Set $S_n' = (S_n - \liminf S_n)/(\limsup S_n - \liminf S_n)$. We observe that $\liminf S_n' = 0$, $\limsup S_n' = 1$, so that we may choose two disjoint subsequences

$$\{S_{\alpha_n}'\} \quad \text{and} \quad \{S_{\beta_n}'\}$$

which converge to 1 and 0 respectively.

We next construct a "replica" $\{S_n^*\}$ of the sequence $\{S_n'\}$ by defining

$$S_n^* = \begin{cases} 1 & \text{if } n = \alpha_k \text{ for some } k. \\ 0 & \text{if } n = \beta_k \text{ for some } k. \\ S_n' & \text{otherwise.} \end{cases}$$

To justify the use of the term replica, we observe that $\{S_n^*\}$ and $\{S_n'\}$ have the same limit points, that $\lim (S_n' - S_n^*) = 0$, and that in general the two sequences behave alike.

Among the terms of $\{S_n^*\}$, there are an infinite number of 1's and of 0's; by the Steinhaus Lemma, for any T, we can choose a subse-

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[1] Hugo Steinhaus, *Kilka słów o uogólnieniu pojecia granicy* (Some remarks on the generalization of limit), Prace Matematyczno-Fizyczne vol. 22 (1911) pp. 121–134 (p. 129).

quence $\{S_{r_n}^*\}$ which is not summable by the regular method T. Since $S_n' = S_n^* + \theta_n$ where θ_n is a null sequence, $\{S_{r_n}'\}$ is not summable T; thus, $\{S_{r_n}\}$ is a subsequence of the original sequence, and is not summable by the method T.

The theorem just proved may be stated in the form of a general Tauberian theorem—admittedly difficult to apply!

THEOREM. *A sequence $\{S_n\}$, summable by a regular matrix method T, is convergent if and only if each of its subsequences is also summable T.*

As a corollary, we may obtain the following result for series:

COROLLARY. *The series $\sum a_n$ is convergent if there exists a regular matrix summability method T, which sums every series formed from the given one by bracketing blocks of terms.*

For, the partial sums of a bracketed series form a subsequence of the partial sums of the original series.

In closing, we note that the restriction to matrix methods is not essential, for the theorem is also true for Abel summability. However, it is apparent that it is *not* true for the Banach-Steinhaus generalized limit, which sums all bounded sequences.[2]

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[2] S. Banach, *Théorie des opérations linéaires*, p. 34.

ON THE SURFACES OF COINCIDENCE

SU-CHENG CHANG

The purpose of this note is to give a characteristic property of the surfaces of coincidence. We show first that there are six and only six asymptotic osculating quadrics Q_j $(j=1, \cdots, 6)$ associated with a point P of a non-ruled analytic surface (M) each of which contains the four consecutive asymptotic tangents $t_{i(j)}$ $(i=1, 2, 3, 4)$ of one system along a curve T through P and on the surface (M). As the curve T must be tangent to a direction of Darboux at P, it is further proved that each quadric Q_j is an osculating quadric of an asymptotic ruled surface R along a curve of Darboux when and only when the surface (M) is a surface of coincidence.

Let the surface (M) be referred to its asymptotic parameters u, v, and let $\{M, M_1, M_2, M_3\}$ be the normal tetrahedron of Cartan, M_1M and M_2M being the two asymptotic tangents and MM_3, M_1M_2 the directrices of Wilczynski. Then the surface (M) is, except for a projective transformation, determined by the following system of equations:

$$
(1)\quad
\begin{cases}
\dfrac{\partial M}{\partial u} = \dfrac{1}{2}\dfrac{\partial \log \gamma}{\partial u} M + M_1, \quad \dfrac{\partial M_1}{\partial u} = B^2 M - \dfrac{1}{2}\dfrac{\partial \log \gamma}{\partial u} M_1 + \beta M_2, \\
\dfrac{\partial M_2}{\partial u} = \dfrac{h_1}{2} M + \dfrac{1}{2}\dfrac{\partial \log \gamma}{\partial u} M_2 + M_3, \\
\dfrac{\partial M_3}{\partial u} = A^2\beta M + \dfrac{1}{2} h_1 M_1 + B^2 M_2 + \dfrac{-1}{2}\dfrac{\partial \log \gamma}{\partial u} M_3; \\
\dfrac{\partial M}{\partial v} = \dfrac{1}{2}\dfrac{\partial \log \beta}{\partial v} M + M_2, \quad \dfrac{\partial M_2}{\partial v} = A^2 M - \dfrac{1}{2}\dfrac{\partial \log \beta}{\partial v} M_2 + \gamma M_1, \\
\dfrac{\partial M_1}{\partial v} = \dfrac{k_1}{2} M + \dfrac{1}{2}\dfrac{\partial \log \beta}{\partial v} M_2 + M_3, \\
\dfrac{\partial M_3}{\partial v} = B^2\gamma M + A^2 M_1 + \dfrac{1}{2} k_1 M_2 - \dfrac{1}{2}\dfrac{\partial \log \beta}{\partial v} M_3;
\end{cases}
$$

where M denotes a point of the surface with coordinates M^i $(i=1, 2, 3, 4)$ and

$$h_1 = \beta\gamma - (\log \beta)'', \qquad k_1 = \beta\gamma - (\log \gamma)''.$$

The integrability conditions of the system (1) are

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$$\frac{\partial A^2}{\partial u} = \frac{h_1}{2}\frac{\partial(\log \beta h_1)}{\partial v}, \qquad \frac{\partial B^2}{\partial v} = \frac{k_1}{2}\frac{\partial(\log \gamma k_1)}{\partial u},$$

(2)

$$A\frac{\partial(A\beta)}{\partial v} = B\frac{\partial(B\gamma)}{\partial u}.$$

Following Godeaux,[1] we denote by $\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots$ the self-polar Laplace sequence in S_5, where $U=|M, M_1|$ and $V=|M, M_2|$ are the images in S_5 of the two asymptotic tangents of the surface (M) at M respectively. It is well known that[2]

$$U^{10} = \beta V, \qquad V^{01} = \gamma U; \tag{3}$$

$$\begin{cases} U_n^{01} = U_{n+1} + U_n(\log \beta h_1 \cdots h_n)^{01}, \\ U_n^{10} = h_n U_{n-1}, \\ h_n = h_{n-1} - (\log \beta h_1 \cdots h_n)^{11}; \end{cases} \tag{4}$$

and

$$\begin{cases} V_n^{10} = V_{n+1} + V_n(\log \gamma k_1 \cdots k_n)^{10}, \\ V_n^{01} = k_n V_{n-1}, \\ k_n = k_{n-1} - (\log \gamma k_1 \cdots k_n)^{11}, \end{cases} \tag{5}$$

where $(\ )^{ij}$ denotes the partial derivative of order $i+j$ formed by differentiating $(\ )$ i times with respect to u and j times with respect to v.

Since seven points in S_5 are always dependent, we find that

$$\begin{aligned} &U_3 + (\log \beta^3 h_1^2 h_2)^{01} U_2 + [(\log \beta h_1)^{02} - 4A^2 \\ &\quad + (\log \beta h_1)^{01}(\log \beta^2 h_1)^{01}]U_1 \\ &\quad - 4A^2(\log A\beta)^{01} U + 4\gamma B^2 V - \gamma(\log \gamma k_1)^{10} V_1 - \gamma V_2 = 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} &V_3 + (\log \gamma^3 k_1^2 k_2)^{10} V_2 + [(\log \gamma k_1)^{20} - 4B^2 \\ &\quad + (\log \gamma k_1)^{10}(\log \gamma^2 k_1)^{10}]V_1 \\ &\quad - 4B^2(\log B\gamma)^{10} V + 4\beta A^2 U - \beta(\log \beta h_1)^{01} U_1 - \beta U_2 = 0. \end{aligned} \tag{7}$$

[1] See Godeaux, *La theorie des surfaces et l'espace réglé*, 1934, Paris. A direct definition of the sequence of Godeaux quadrics has been given by the present author. See S. C. Chang, *Some theorems on ruled surfaces*, in the press of Science Records, Academia Sinica.

[2] L. Godeaux, loc. cit.

If four consecutive asymptotic u-tangents (v=const.) belong to a regulus, then there is a plane in S_5 which intersects the surface (U) at four consecutive points. In other words: U, dU, d^2U, d^3U should be coplanar.

By means of (3), (4), (5) and (6) it is clear that the three points

$$\beta V du + U_1 dv, \tag{8}$$

$$V[(\beta^{10} + \beta(\log\gamma)^{10})(du)^2 + \beta^{01}dudv + \beta d^2u] + \beta V_1(du)^2 + U_1 d^2v + [U_2 + U_1(\log\beta h_1)^{01}](dv)^2 \tag{9}$$

and

$$[U_3 - 4A^2(\log A\beta)^{01}U](dv)^3 + MU_1 + \beta V_2(du)^3 + NV + V_1[\beta(\log\gamma^2 k_1)^{10}(du)^3 + 2\beta^{10}(du)^3 + 2\beta^{01}dv(du)^2 + 3\beta dud^2u] + U_2[(\log\beta^2 h_1^2 h_2)^{01}(dv)^3 + 3dvd^2v] \tag{10}$$

are collinear, where

$$M = [(\log\beta h_1)^{01}]^2(dv)^3 + d^3v + h_1 du(dv)^2 + (\log\beta h_1)^{02}(dv)^3 + 3(\log\beta h_1)^{01}d^2vdv,$$

$$N = 3\beta^{10}dud^2u + (du)^3[\beta^{20} + \beta((\log\gamma)^{10})^2 + \beta(\log\gamma)^{20} + 2\beta^{10}(\log\gamma)^{10}] + (du)^2dv[2\beta^{11} + \beta(\log\gamma)^{11} + 2\beta^{01}(\log\gamma)^{10} + \beta k_1] + \beta^{02}du(dv)^2 + 2\beta^{01}d^2udv + \beta^{01}dud^2v + \beta d^3u + 3\beta(\log\gamma)^{10}d^2udu.$$

From (6), (8), (9) and (10) it follows that the coefficient of V_2 in (10) must vanish, so that

$$\beta(du)^3 + \gamma(dv)^3 = 0. \tag{11}$$

Moreover, we demand that the coefficients of U_2, V_1 in (9) should be proportional to those in (10), and therefore that

$$dud^2v - dvd^2u - (1/3)dudv[(\log\beta^2\gamma)^{10}du + (\log\beta^3)^{01}dv] = 0. \tag{12}$$

Differentiation of (11) shows that

$$3(dud^2v - dvd^2u) + (du)^2dv(\log(\gamma/\beta))^{10} + du(dv)^2(\log(\gamma/\beta))^{01} = 0. \tag{13}$$

If (12) and (13) coincide with each other for any direction of Darboux, then

$$(\log\beta\gamma^2)^{10} = 0, \qquad (\log\gamma\beta^2)^{01} = 0. \tag{14}$$

Thus we may take

$$\beta = 1, \qquad \gamma = 1, \tag{15}$$

and accordingly,

$$N = d^3u + (du)^2dv, \qquad M = d^3v + du(dv)^2.$$

Hence the surface in consideration must be a surface of coincidence.

We inquire now whether the asymptotic tangents of some one system along a curve on a surface belong to a regulus. It is seen from (11) that the curve is necessarily a Darboux curve. In virtue of (11) and (15) we obtain a point

$$V[d^3u + (du)^2dv + (-4B^2)(dv)^3 - 3d^2vd^2u/dv] + U_1[d^3v + du(dv)^2 + 4A^2(du)^3 - 3(d^2v)^2/dv].$$

This is collinear with the points (9) and (10) and coincides with the point (8) when

$$dv[d^3u + (du)^2dv - 4B^2(dv)^3 - 3d^2vd^2u/dv] - du[d^3v + du(dv)^2 + 4A^2(dv)^3 - 3(d^2v)^2/dv] = 0. \tag{16}$$

From (13), (15) and (16) we have

$$A^2 = 0, \qquad B^2 = 0. \tag{17}$$

A reference to (15) and (17) shows that *the surface must be* $xyz=1$ *or one of its projective transforms.*

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THE CHARACTERISTICS OF ASYMPTOTIC OSCULATING QUADRICS OF A CURVE ON A SURFACE

BUCHIN SU[1]

1. **Introduction.** Lane[2] has derived the equations of asymptotic osculating quadrics of a curve C on a surface S, employing Wilczynski's notation and deducing some of their fundamental properties. We shall now investigate the characteristics of these quadrics along C. For the sake of convenience the normal tetrahedron of Cartan[3] is utilized throughout this note. In terms of the local coordinates with respect to this tetrahedron we have expressed simply the equations of asymptotic osculating quadrics and their characteristics and therefrom obtained new configurations projectively connected with a surface.

2. **The normal tetrahedron of Cartan.** Let us consider the directrices of Wilczynski at a generic point M of a non-ruled surface S; the first of them intersects the quadric of Lie at another point M_3 and the second meets the asymptotic tangents at the points M_1 and M_2. The tetrahedron $\{MM_1M_2M_3\}$ is known as the normal tetrahedron of Cartan. If we denote, for simplicity, the corresponding projective coordinates of these points M, M_1, M_2, M_3 by the same notations, then they are solutions of the following system of differential equations:[4]

$$
\begin{aligned}
\frac{\partial M}{\partial u} &= \frac{1}{2} M \frac{\partial}{\partial u} \log \gamma + M_1, \\
\frac{\partial M_1}{\partial u} &= B^2 M - \frac{1}{2} M_1 \frac{\partial}{\partial u} \log \gamma + \beta M_2, \\
\frac{\partial M_2}{\partial u} &= KM + \frac{1}{2} M_2 \frac{\partial}{\partial u} \log \gamma + M_3, \\
\frac{\partial M_3}{\partial u} &= A^2 \beta M + KM_1 + B^2 M_2 - \frac{1}{2} M_3 \frac{\partial}{\partial u} \log \gamma;
\end{aligned}
\tag{1}
$$

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[2] E. P. Lane, *The asymptotic osculating quadrics of a curve on a surface*, Bull. Amer. Math. Soc. vol. 33 (1927) pp. 195–200.

[3] Cf., for example, Fubini and Čech, *Introduction à la géométrie projective différentielle des surfaces*, Paris, 1931, p. 236.

[4] These were first used in this form by S. Finikoff, *Sur les couples de surfaces dont les asymptotiques se correspondent et qui, aux points correspondants, ont les mêmes directrices de Wilczynski*, C. R. Acad. Sci. Paris vol. 197 (1933) pp. 883–885.

$$
\begin{aligned}
&\frac{\partial M}{\partial v} = \frac{1}{2} M \frac{\partial}{\partial v} \log \beta + M_2, \\
&\frac{\partial M_1}{\partial v} = \overline{K} M + \frac{1}{2} M_1 \frac{\partial}{\partial v} \log \beta + M_3, \\
&\frac{\partial M_2}{\partial v} = A^2 M + \gamma M_1 - \frac{1}{2} M_2 \frac{\partial}{\partial v} \log \beta, \\
&\frac{\partial M_3}{\partial v} = B^2 \gamma M + A^2 M_1 + \overline{K} M_2 - \frac{1}{2} M_3 \frac{\partial}{\partial v} \log \beta,
\end{aligned} \tag{1}
$$

where the components of displacements of the tetrahedron β, γ, A, B, K, $\overline{K}$ satisfy the integrability conditions

$$
\begin{gathered}
2K = \beta\gamma - \frac{\partial^2}{\partial u \partial v} \log \beta, \qquad 2\overline{K} = \beta\gamma - \frac{\partial^2}{\partial u \partial v} \log \gamma, \\
\frac{\partial (A^2)}{\partial u} = K \frac{\partial}{\partial v} \log (K\beta), \qquad \frac{\partial (B^2)}{\partial v} = \overline{K} \frac{\partial}{\partial u} \log (\overline{K}\gamma), \\
A \frac{\partial (A\beta)}{\partial v} = B \frac{\partial (B\gamma)}{\partial u}.
\end{gathered} \tag{2}
$$

Any point P in space can now be represented by the local coordinates (y_1, y_2, y_3, y_4) with respect to the tetrahedron $\{MM_1M_2M_3\}$,

$$
P = y_1 M + y_2 M_1 + y_3 M_2 + y_4 M_3. \tag{3}
$$

From (1) we easily show that the conditions of immovability for P are

$$
\begin{aligned}
&\frac{\partial y_1}{\partial u} = -\frac{1}{2}\left(\frac{\partial}{\partial u} \log \gamma\right) y_1 - B^2 y_2 \qquad\qquad - K y_3 \qquad\qquad - A^2 \beta y_4, \\
&\frac{\partial y_2}{\partial u} = \qquad - y_1 + \frac{1}{2}\left(\frac{\partial}{\partial u} \log \gamma\right) y_2 \qquad * \qquad\qquad - K y_4, \\
&\frac{\partial y_3}{\partial u} = \quad * \qquad\qquad - \beta y_2 \qquad\qquad - \frac{1}{2}\left(\frac{\partial}{\partial u} \log \gamma\right) y_3 - B^2 y_4, \\
&\frac{\partial y_4}{\partial u} = \quad * \qquad\qquad * \qquad\qquad - y_3 \qquad\qquad + \frac{1}{2}\left(\frac{\partial}{\partial u} \log \gamma\right) y_4; \\
&\frac{\partial y_1}{\partial v} = -\frac{1}{2}\left(\frac{\partial}{\partial v} \log \beta\right) y_1 - \overline{K} y_2 \qquad\qquad - A^2 y_3 \qquad\qquad - B^2 \gamma y_4, \\
&\frac{\partial y_2}{\partial v} = \quad * \qquad\qquad - \frac{1}{2}\left(\frac{\partial}{\partial v} \log \beta\right) y_2 - \gamma y_3 \qquad\qquad - A^2 y_4, \\
&\frac{\partial y_3}{\partial v} = - y_1 \qquad\qquad * \qquad\qquad + \frac{1}{2}\left(\frac{\partial}{\partial v} \log \beta\right) y_2 - \overline{K} y_4, \\
&\frac{\partial y_4}{\partial v} = \quad * \qquad\qquad - y_2 \qquad\qquad * \qquad\qquad + \frac{1}{2}\left(\frac{\partial}{\partial v} \log \beta\right) y_4.
\end{aligned} \tag{4}
$$

3. **Equations.** Suppose that a curve C on S is given by the equation

$$u = u(v). \tag{5}$$

The asymptotic ruled surface R_u generated by the asymptotic u-tangents of S along C is given by

$$P = M + \rho M_1, \tag{6}$$

where ρ and v are independent parameters, the variable u in M and M_1 being given by (5).

In virtue of (1) there is no difficulty in calculating the partial derivatives of P with respect to ρ and v. Denoting $du/dv, d^2u/dv^2, d^3u/dv^3$ by u', u'', u''' respectively we have

$$\begin{aligned}
P_\rho &= M_1, \\
P_v &= \left\{\frac{1}{2}u'\frac{\partial}{\partial u}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta + \rho(B^2u' + \overline{K})\right\}M \\
&\quad + \left\{u' + \frac{1}{2}\rho\left(-u'\frac{\partial}{\partial u}\log\gamma + \frac{\partial}{\partial v}\log\beta\right)\right\}M_1 \\
&\quad + (1 + \rho\beta u')M_2 + \rho M_3, \\
P_{v\rho} &= (B^2u' + \overline{K})M + \frac{1}{2}\left(-u'\frac{\partial}{\partial u}\log\gamma + \frac{\partial}{\partial v}\log\beta\right)M_1 \\
&\quad + \beta u' M_2 + M_3, \\
P_{vv} &= (*)M + (*)M_1 \\
&\quad + \left\{u'\frac{\partial}{\partial u}\log\gamma + \beta u'^2 + \rho(2B^2u' + 2\overline{K} \right. \\
&\qquad \left. + \beta u'' + \beta_u u'^2 + \beta_v u')\right\}M_2 \\
&\quad + 2\left\{u' + \rho\left(-u'\frac{\partial}{\partial u}\log\gamma + \beta u'^2\right)\right\}M_3,
\end{aligned} \tag{7}$$

so that the differential equation of the curved asymptotics of[5] R_u,

$$2(PP_\rho P_v P_{v\rho})d\rho + (PP_\rho P_v P_{vv})dv = 0,$$

becomes

$$\frac{d\rho}{dv} = a\rho^2 + b\rho - u', \tag{8}$$

[5] The parenthesis denotes a determinant of the four points there inclosed.

where we have placed

$$
\begin{aligned}
a &= \overline{K} + B^2u' + \frac{1}{2}\beta u'' + \frac{1}{2}\beta u'^2\frac{\partial}{\partial u}\log(\beta\gamma) \\
&\quad + \frac{1}{2}\beta u'\frac{\partial}{\partial v}\log\beta - \frac{1}{2}\beta^2u'^3, \qquad (9)\\
b &= u'\frac{\partial}{\partial u}\log\gamma - \beta u'^2.
\end{aligned}
$$

The osculating quadric Q_u or R_u along its generator MM_1 is the asymptotic osculating quadric of one family, and a second asymptotic osculating quadric Q_v of C at M is obtained by using the other family of asymptotics of S. It is clear, therefore, that a generic point Z on Q_u must be represented by

$$
Z = P + \sigma\frac{dP}{dv} = M + \rho M_1 + \sigma\left(P_v + P_\rho\frac{d\rho}{dv}\right), \qquad (10)
$$

σ being another parameter.

Substitution of (7) and (8) in (10) gives the local coordinates of the point Z, namely,

$$
\begin{aligned}
y_1 &= 1 + \sigma\left\{\frac{1}{2}u'\frac{\partial}{\partial u}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta + \rho(B^2u' + \overline{K})\right\}, \\
y_2 &= \rho + \rho\sigma\left\{\frac{1}{2}u'\frac{\partial}{\partial v}\log\gamma + \frac{1}{2}\frac{\partial}{\partial v}\log\beta - \beta u'^2\right\} + a\rho^2\sigma, \qquad (11)\\
y_3 &= (1 + \rho\beta u')\sigma, \\
y_4 &= \rho\sigma.
\end{aligned}
$$

Thus we obtain *the equation of the first asymptotic osculating quadric Q_u of C at M*:

$$
\begin{aligned}
&y_1y_4 - y_2y_3 + \beta u'y_2y_4 - \beta u'^2y_3y_4 \\
&\quad + \frac{1}{2}\beta\left\{u'' + \beta u'^3 + u'^2\frac{\partial}{\partial u}\log(\beta\gamma) + u'\frac{\partial}{\partial v}\log\beta\right\}y_4^2 = 0. \qquad (12)
\end{aligned}
$$

The equation of the second asymptotic osculating quadric Q_v of C at M is

$$
\begin{aligned}
&u'^2(y_1y_4 - y_2y_3) - \gamma u'y_2y_4 + \gamma u'^2y_3y_4 \\
&\quad + \frac{1}{2}\gamma\left\{-u'' + \gamma + u'\frac{\partial}{\partial v}\log(\beta\gamma) + u'^2\frac{\partial}{\partial u}\log\gamma\right\}y_4^2 = 0. \qquad (13)
\end{aligned}
$$

4. **Characteristics.** The asymptotic osculating quadrics Q_u (or Q_v) at M and its consecutive point M' on the curve C intersect each other at the asymptotic tangent u (or v) and two other lines, which are called the characteristic of Q_u (or Q_v) along C. In order to derive the equations of the characteristics of Q_u we have to set the derivative along C of the left-hand member of (12) equal to zero, remembering that the derivatives of y's should be determined by (4). A simple calculation, which we shall omit here, suffices to demonstrate that the equations of the characteristic in consideration are (12) and

$$C_{33}y_3^2 - C_{34}y_3y_4 + C_{44}y_4^2 = 0, \tag{14}$$

where we have placed

$$C_{33} = \beta u'^2 + \gamma, \tag{15}$$

$$C_{34} = \beta u'\left\{3u'' + \beta u'^2 + \left(\frac{\partial}{\partial u}\log(\beta^2\gamma)\right)u'^2 + 3\left(\frac{\partial}{\partial v}\log\beta\right)u' + \gamma\right\}, \tag{16}$$

$$\begin{aligned} C_{44} = {} & -A^2\beta u' - B^2\gamma + \beta u'^2(B^2u + \overline{K}) - \beta u'(Ku' + A^2) \\ & + \frac{1}{2}\beta\left\{u'' + \beta u'^3 + \left(\frac{\partial}{\partial u}\log(\beta\gamma)\right)u'^2 + \left(\frac{\partial}{\partial u}\log\beta\right)u'\right\}\left\{u'\frac{\partial}{\partial u}\log\gamma + \frac{\partial}{\partial v}\log\beta\right\} \\ & + \frac{1}{2}\beta\left\{u''' + \left(\frac{\partial}{\partial u}\log(\beta^2\gamma^2)\right)u'u'' + 2\left(\frac{\partial}{\partial v}\log\beta\right)u'' + 3\beta^2u'^2u''\right. \\ & \left. + 2\left(\frac{\partial}{\partial u}\log\beta\right)u'^4 + \left(2\frac{\partial}{\partial v}\log\beta + \frac{\partial^2}{\partial u^2}\log(\beta\gamma) + \frac{\partial}{\partial u}\log(\beta\gamma)\frac{\partial}{\partial u}\log\beta\right)u'^3\right. \\ & \left. + \left(\frac{\partial^2}{\partial u\partial v}\log(\beta^2\gamma) + 2\frac{\partial}{\partial u}\log(\beta\gamma)\frac{\partial}{\partial v}\log\beta\right)u'^2 + \left(\frac{\partial^2}{\partial v^2}\log\beta + \left(\frac{\partial}{\partial v}\log\beta\right)^2\right)u'\right\}. \end{aligned} \tag{17}$$

From (14) it follows that the characteristic of Q_u along C consists of two straight lines intersecting the asymptotic u-tangent and the latter counted twice.

In a similar way we obtain another pair of lines intersecting the asymptotic tangent MM_2 as the second quadric moves along C. These four lines furnish obviously a generalization of the well known quadrilateral of Demoulin.

By means of (14) we can deduce some remarkable results.[6]

(I) *If the asymptotic tangent MM_2 is a part of the characteristic of the asymptotic osculating quadric Q_u of C at M, then C must be tangent to a Darboux curve at M, and conversely.*

The necessary and sufficient condition for this is $C_{33}=0$, namely,

$$\beta u'^3 + \gamma = 0, \tag{18}$$

which represents the Darboux directions of the surface at M.

(II) *In order that the characteristic of the first and second asymptotic osculating quadrics along every Darboux curve of a surface S should decompose into two asymptotic tangents, each being counted twice, the necessary and sufficient condition is that S be a surface of coincidence.*

In fact, we have $C_{33}=0$, $C_{34}=0$, so that (14) becomes $y_4^2=0$. In consequence, the characteristic is given by the equations

$$y_4^2 = 0, \qquad y_2 y_3 = 0.$$

For a Darboux curve we have

$$\beta u'^3 + \gamma = 0, \quad u'' = -\frac{1}{3}u'^2\frac{\partial}{\partial u}\log(\gamma/\beta) - \frac{1}{3}u'\frac{\partial}{\partial v}\log(\gamma/\beta). \tag{19}$$

Substituting these in the condition $C_{24}=0$, namely,

$$3u'' + \beta u'^3 + u'^2\frac{\partial}{\partial u}\log(\beta^2\gamma) + 3u'\frac{\partial}{\partial v}\log\beta + \gamma = 0, \tag{20}$$

and taking account of the fact that this equation must hold for the three Darboux curves, we arrive at

$$\frac{\partial}{\partial v}\log\beta = 0, \qquad \frac{\partial}{\partial v}\log\gamma = 0,$$

that is, both β and γ are functions of u alone.

[6] Cf. S. C. Chang, *On the surfaces of coincidence*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 900–903.

Since the same condition also holds for the second asymptotic osculating quadrics, we have that β and γ must be functions of v alone, and therefore that they are constants. Hence the surface in question is a surface of coincidence.

(III) *If the characteristics of the quadrics Q_u and Q_v along every Darboux curve of a surface S are indeterminate, that is, the asymptotic osculating quadrics along every Darboux curve are stationary, then S must be projectively equivalent to the cubic surface $xyz=1$.*

For we have in this case

$$C_{33} = 0, \qquad C_{34} = 0, \qquad C_{44} = 0.$$

We may put $\beta=\gamma=1$, which gives

$$A^2 = 0, \qquad B^2 = 0.$$

The equation to the surface may easily be found by integrating the system (1).

5. **Associate directrices.** We come now to demonstrate that a certain pair of covariant lines can be constructed by means of the asymptotic osculating quadrics. Consider, for instance, the first quadric Q_u given by (12). As was shown before, there exists an element of the second order, E_2, corresponding to each Darboux tangent, such that the characteristic of Q_u at the point M decomposes into the two asymptotic tangents MM_1 and MM_2, each being counted twice. We have therefore a plane containing this E_2. In order to find the equation of this plane we merely have to substitute the value of u'' given by (20) in the equation of the osculating plane of C at M:

$$2u'(x - u'y) - \left(u'' + u'^2\frac{\partial}{\partial u}\log(\beta\gamma) + \gamma - u'\frac{\partial}{\partial v}\log(\beta\gamma) - \beta u'^3\right)z = 0, \tag{21}$$

where x, y, z denote the nonhomogeneneous coordinates of a point with respect to the tetrahedron of Fubini at M. The result of carrying out the computation is

$$2u'(x - u'y) - \{\phi u'^2/3 - \psi u' + 2\gamma\}z = 0, \tag{22}$$

where

$$u' = -\epsilon(\gamma/\beta)^{1/3} \qquad (\epsilon \neq 1, \epsilon^3 = 1), \tag{23}$$

$$\phi = \frac{\partial}{\partial u}\log(\beta\gamma^2), \qquad \psi = \frac{\partial}{\partial v}\log(\beta^2\gamma). \tag{24}$$

The three planes (22) corresponding to the Darboux tangents at M form a trihedron, and the polar of the tangent plane of the surface at M with respect to this is

$$x + \psi z/2 = 0, \qquad y + \phi z/6 = 0. \tag{25}$$

If the second quadric Q_v is used instead of Q_u, a second covariant line is found to be

$$x + \psi z/6 = 0, \qquad y + \phi z/2 = 0. \tag{26}$$

These lines are analogous to the directrices of Sullivan[7] and will be called *the associate directrices* of the surface at M. The first of them, given by (25), is the intersection of the plane containing the asymptotic v-tangent and the first directrix of Wilczynski and the plane containing the asymptotic u-tangent and the first principal ray $c(-1/6)$ of Fubini. A similar construction is obtained for the second associate directrix. It shall be noted that the asymptotic u-tangent, the first directrix of Sullivan, the first associate directrix and the first directrix of Wilczynski are coplanar and the double ratio of them in this order is equal to -3.

There are two planes through each Darboux tangent at M, one being (22) and the other for Q_v:

$$x - u'y + (1/2)\{\psi - \phi u'/3 + 2\beta u'^2\}z = 0. \tag{27}$$

The harmonic conjugate of the tangent plane of the surface with respect to them osculates the corresponding Darboux curve at M, as we can easily show from the equations (19) and (21).

6. **Associate cones.** As was shown by Lane,[8] the directrices of Sullivan are related to a certain pair of cones of the third class enveloped by the osculating planes at M of the extremals of certain integrals. We shall here prove that analogous cones also exist in the case of associate directrices.

Let us consider again the characteristic of the first quadric Q_u along a curve C on the surface S. For a given non-Darboux direction equation (14) shows that we can always adjust an element of the second order, E_2, to the curve C of the given direction at M, such that the planes through the asymptotic u-tangent and the characteristic lines are harmonic conjugate to the tangent plane and the plane through the first directrix of Wilczynski. The plane containing this E_2 cor-

[7] C. T. Sullivan, *Scroll directrix curves*, Trans. Amer. Math. Soc. vol. 16 (1915) pp. 199–214.

[8] E. P. Lane, *Le direttrici di Sullivan*, Bollettino della Unione Matematica Italiana vol. 5 (1926) pp. 214–215.

responding to every non-Darboux direction at M envelopes a cone of the third class

$$\gamma u_1^3 + 2\beta u_2^3 + 3(\psi u_1/2 + \phi u_2/6 - u_3)u_1u_2 = 0, \tag{28}$$

where u_1, u_2, u_3 denote the coordinates of a plane through M. The three cuspidal planes of this cone evidently pass through the first associate directrix.

In a similar way a second cone of the third class

$$2\gamma u_1^3 + \beta u_2^3 + 3(\psi u_1/6 + \phi u_2/2 - u_3)u_1u_2 = 0 \tag{29}$$

is obtained.

In virtue of these cones we can further construct a third cone of the third class and a new canonical ray at the point M of the surface.

For this purpose, we draw the tangent planes of the cones (28) and (29) through a non-Darboux tangent at M and construct the harmonic conjugate of the tangent plane of the surface with respect to them. The equation of this plane is found to be

$$2u'(x - u'y) + (\beta u'^3 - \gamma + \phi u'^2/3 - \psi u'/3)z = 0. \tag{30}$$

It is easily seen that the three planes corresponding to the three Segre tangents at M are concurrent in the canonical ray $c(1/6)$:

$$x - \psi z/6 = 0, \qquad y - \phi z/6 = 0. \tag{31}$$

This line and the first principal ray of Fubini harmonically separate the canonical tangent and the projective normal of the surface at M.

The plane (30) also envelopes a cone of the third class

$$\gamma u_1^3 + \beta u_2^3 - 2(\psi u_1/6 + \phi u_2/6 + u_3)u_1u_2 = 0 \tag{32}$$

and the three cuspidal planes are concurrent in the ray $c(1/6)$.

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CONVERGENCE REGIONS FOR THE GENERAL CONTINUED FRACTION

W. J. THRON

The purpose of this note is to prove convergence region theorems for continued fractions

$$\text{(1)} \qquad \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\cdots,$$

where the a_n and b_n are complex numbers. As an application we obtain a new theorem for the associated continued fraction

$$\text{(2)} \qquad \frac{a_1}{b_1+z+}\,\frac{a_2}{b_2+z+}\,\cdots.$$

This continued fraction plays a rôle in the moment problem and is also otherwise of interest as a means of representing analytic functions. For details see Perron [4, chapters 8 and 9].[1]

Recently a continued fraction very similar to (2) has been investigated by Hellinger and Wall [1]. They chose the name "J-fraction." A J-fraction is a continued fraction of the form

$$\text{(2)}' \qquad \frac{1}{b_1+z-}\,\frac{c_1^2}{b_2+z-}\,\frac{c_2^2}{b_3+z-}\,\cdots.$$

Our result for J-fractions is stated in Theorem C.

The results of this note are closely related to an as yet unpublished work of Wall and Wetzel on "positive definite J-fractions." In particular Theorem C seems to be contained in a theorem of theirs.

In what follows we shall denote by $H(b, \gamma)$ the half-plane (including the boundary) defined by the relation $z \in H(b, \gamma)$ if $\Re(ze^{-i\gamma}) \geqq b$. For the open half-plane we shall use the notation $H_0(b, \gamma)$. It is clear from the context that b is a real number. Further for $a > 0$, $P(a, \gamma)$ shall be the parabolic region (including the boundary) bounded by the curve

$$\rho \leqq \frac{a^2/2}{1-\cos(\theta-2\gamma)}.$$

For $a = 0$, $P(a, \gamma)$ is to be the totality of points $r \cdot e^{i\gamma}$, $r \geqq 0$.

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[1] Numbers in brackets refer to the references listed at the end of the paper.

We are now in a position to state our theorems.

THEOREM A. *The continued fraction* (1) *converges if all the* a_n *lie in a bounded part of the set* $P(a, \gamma)$, $a \geqq 0$, *and if all* b_n *lie in the half-plane* $H(a+\epsilon, \gamma)$, *where* ϵ *is an arbitrary small positive number.*

Even though its statement is somewhat involved it seems worth while to state the following theorem too, as it is a generalization of the parabola theorems[2] for the continued fractions with $b_n = 1$.

THEOREM A'. *In the continued fraction* (1) *let all* b_n *lie in the angular opening*

$$|\arg e^{-i\gamma}(b_n - 1)| < \pi/2 - \epsilon,$$

where $-\pi/2 < \gamma < \pi/2$ *and* ϵ *is an arbitrary small positive number. Further let all the* a_n *lie in the parabolic region* $P(\cos\gamma, \gamma)$. *Then the continued fraction* (1) *converges if and only if* $\sum |b_n d_n| = \infty$, *where* $d_1 = 1/a_1$, $d_n = 1/d_{n-1}a_n$.

It is well known that the last condition of the theorem is satisfied if $\liminf a_n < \infty$. We note that neither of these two theorems is contained in the other. Both theorems could be used to obtain new "multiple" convergence regions for continued fractions $1+K(a_n/1)$. For both of these theorems there are corresponding theorems for associated continued fractions and J-fractions. We only state the theorems corresponding to Theorem A.

THEOREM B. *If real numbers* $a \geqq 0$, $M > 0$, b *and* γ *exist such that for all* $n \geqq 1$, $a_n \in P(a, \gamma)$, $|a_n| < M$ *and* $b_n \in H(b, \gamma)$ *then the associated continued fraction* (2) *converges uniformly and hence to a holomorphic function of the complex variable* z *in every closed region contained in the half-plane* $H_0(a-b, \gamma)$.

It is easily seen that if $a_n \in P(a, 0)$ then $c_n = a_n^{1/2}$ satisfies the relation $|\Im(c_n)| \leqq a/2$ and conversely. Hence the following theorem for J-fractions is an immediate consequence of Theorem B with $\gamma = 0$.

THEOREM C. *If real numbers* $a \geqq 0$, $M > 0$ *and* b *exist such that for all* $n \geqq 1$, $|\Im(c_n)| \leqq a/2$, $|c_n| < M$ *and* $\Im(b_n) \geqq b$ *then the J-fraction* (2)' *converges uniformly and hence to a holomorphic function of the complex variable* z *in every closed region contained in the half-plane* $\Im(z) > a-b$.

The theorems are based on the following lemma.

LEMMA. *All the approximants of the continued fraction* (1) *lie in the half-plane* $H(-b/2, \gamma)$ *if for all* $n \geqq 1$, $a_n \in P(b, \gamma)$ *and* $b_n \in H(b, \gamma)$, *where* $b \geqq 0$.

[2] See Leighton and Thron [2] and Paydon and Wall [3].

The proof of the lemma is by induction. To perform the induction we require the following two relations: for an arbitrary choice of $a_n \in P(b, \gamma)$, $b_n \in H(b, \gamma)$ and $v \in H(-b/2, \gamma)$ the two relations

$$a_n/b_n \in H(-b/2, \gamma) \tag{3}$$

and

$$\frac{a_n}{b_n + v} \in H(-b/2, \gamma) \tag{4}$$

must be satisfied. We note that (3) is a consequence of (4) as $v=0 \in H(-b/2, \gamma)$. Further as b_n and v vary independently over their respective regions their sum $b+v$ varies over the half-plane $H(b/2, \gamma)$. It is then easily seen that relation (4) is satisfied if $P(b, \gamma)$ is the part common to all the regions $c \cdot H(-b/2, \gamma)$, ($z \in c \cdot H(-b/2, \gamma)$ if $z = c \cdot v$, where $v \in H(-b/2, \gamma)$) where c varies over $H(b/2, \gamma)$. For the proof of this fact we refer the reader to [2, §2], where a very similar fact is proved in detail.

We now proceed to the proof of Theorem B. Under the conditions of that theorem $b_n + z \in H_0(a, \gamma)$ for all $n \geqq 1$. According to the lemma all the approximants of the continued fraction (2) then lie in the half-plane $H(-a/2, \gamma)$. All approximants are finite. This is seen as follows: In order that an approximant be infinite it is necessary that

$$-(b_1 + z) = \frac{a_2}{b_2 + z} + \cdots + \frac{a_n}{b_n + z}.$$

This is impossible as the regions $-H_0(a, \gamma)$ and $H(a/2, \gamma)$ have no point in common.

The approximants of (2) are rational functions of z; for z in $H_0(a-b, \gamma)$ they are finite and do not take on certain values (more than two), hence by Montel's Theorem the sequence of approximants forms a normal family of holomorphic functions for z in $H_0(a-b, \gamma)$.

As the $|a_n|$ are bounded there exists an M such that for $|z| > M$, $z \in H_0(a-b, \gamma)$,

$$|b_n + z| > |a_n| + 1$$

for all n. For these values of z the continued fraction converges by Pringsheim's Theorem [4, p. 254]. It then follows from the generalized Stieltjes-Vitali Theorem that the continued fraction converges uniformly in every closed region contained in the half-plane $H_0(a-b, \gamma)$. This completes the proof of Theorem B.

Theorem A follows from Theorem B if we set $a=b$ and let $z = \epsilon e^{i\gamma}$.

For the proof of Theorem A′ we consider the continued fraction

$$\frac{a_1}{1+d_1e^{i\theta_1 z}}+\frac{a_2}{1+d_2e^{i\theta_2 z}}+\cdots, \tag{5}$$

where $a_n \in P(\cos\gamma, \gamma)$, $d_n \geqq 0$ and $|\theta_n - \gamma| < \pi/2 - \epsilon$ for all $n \geqq 1$.

Under these conditions the approximants of (5) form a normal family of holomorphic functions for z in the region D defined by

$$-\delta < \Re(z) < 1+\delta,$$

where δ is positive and depends on γ and ϵ. The proof of this fact is similar to the proof used in the previous case. For $\Re(z)=0$ the continued fraction converges. In this case the partial denominators are real and greater than or equal to 1 and hence (5) can be transformed into a continued fraction of the form

$$\frac{g_1}{1+}\,\frac{g_2}{1+}\cdots,$$

where all $g_n \in P(\cos\gamma, \gamma)$. The convergence of this continued fraction follows from the parabola theorems (we are assuming that all conditions of Theorem A′ are satisfied). The Stieltjes-Vitali Theorem then insures the convergence of (5) for all z in D. If we recall that $\sum|c_n| = \infty$ is a necessary condition for convergence we have Theorem A′ by setting $z=1$ in (5).

References

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2. W. Leighton and W. J. Thron, *Continued fractions with complex elements*, Duke Math. J. vol. 9 (1942) pp. 763–772.

3. J. F. Paydon and H. S. Wall, *The continued fraction as a sequence of linear transformations*, Duke Math. J. vol. 9 (1942) pp. 360–372.

4. O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig, 1929.

The Rice Institute

A THEOREM ON GENERALIZED DERIVATIVES

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1. **General remarks.** Suppose that a function $f(x)$, defined in the neighborhood of a point x_0, satisfies a relation

$$f(x_0 + t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2/2! + \cdots + \alpha_{k-1} t^{k-1}/(k-1)! + \omega_k(t) t^k/k!, \tag{1.1}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_{k-1}$ are independent of t, and the expression $\omega_k(t) = \omega(x_0, t)$ approaches a finite limit α_k as t tends to 0. The function f is then said to possess a kth generalized derivative at the point x_0, and α_k is the value of that derivative. Instead of α_k we shall write $D_k f(x_0)$. It is clear that the existence of $D_k f(x_0)$ implies that of $D_{k-1} f(x_0)$.

The existence of $D_0 f(x_0)$ is simply continuity of the function f at the point $x = x_0$. For $k = 1$ the definition of $D_k f(x_0)$ is equivalent to that of the ordinary derivative $f^{(k)}(x_0)$. No such equivalence exists for higher values of k, for then the existence of $D_k f(x_0)$ does not even imply continuity of f for $x \neq x_0$. However, if $f^{(k)}(x_0)$ exists and is finite, then $D_k f(x_0)$ also exists and is equal to $f^{(k)}(x_0)$.

It is a classical result of Fatou that, if a function $f(x)$ is everywhere continuous and, say, of period 2π, then the integral

$$\int_0^\pi \frac{f(x+t) - f(x-t)}{t}\, dt = \lim_{\delta \to +0} \int_\delta^\pi \frac{f(x+t) - f(x-t)}{t}\, dt$$

exists for almost every x.[1] This integral may also be written

$$\int_0^\pi \frac{\omega_0(x, t) - \omega_0(x, -t)}{t}\, dt \tag{1.2}$$

or

$$\int_0^\pi \frac{\epsilon_0(x, t) - \epsilon_0(x, -t)}{t}\, dt \tag{1.3}$$

if for any x for which $D_k f(x)$ exists we introduce the notation

$$\omega_k(x, t) = D_k f(x) + \epsilon_k(x, t)$$

(so that $\epsilon(x, t)$ tends to 0 with t, for x fixed).

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[1] For this and more general results, see, for example, the author's *Trigonometric Series*, Warsaw, 1935, chap. 7.

The convergence of the integral (1.3) may be due either to the smallness of the numerator $\epsilon_0(x, t) - \epsilon_0(x, -t)$ (which, anyway, tends to 0) or to the interference of positive and negative values of the integrand. In general, it is the second explanation which is right. For there exist continuous functions $f(x)$ such that the integral

$$\int_0^\pi \frac{|\epsilon_0(x, t) - \epsilon_0(x, -t)|}{t}\, dt \tag{1.4}$$

is infinite for every value of x.[2] For what follows it is of interest to observe that there exist continuous functions f such that the integral

$$\int_0^\pi \frac{|\epsilon_0(x, t) - \epsilon_0(x, -t)|^r}{t}\, dt \tag{1.5}$$

is infinite for every x and for every positive number r, however large.[3] Since $\epsilon_0(x, t) - \epsilon_0(x, -t)$ tends to 0 with t, the divergence of (1.5) implies the divergence of the integral with any exponent smaller than r.

The result of Fatou has been generalized in several ways. In particular, Plessner showed that, if

(i) $f(x)$ is integrable over $(0, 2\pi)$ and is of period 2π;

(ii) $D_k f(x)$ exists for every point x of a set E of positive measure,

then the integral

$$\int_0^\pi \frac{\omega_k(x, t) - \omega_k(x, -t)}{t}\, dt = \int_0^\pi \frac{\epsilon_k(x, t) - \epsilon_k(x, -t)}{t}\, dt = \lim_{\delta \to +0} \int_\delta^\pi \tag{1.6}$$

exists for almost every $x \in E$.[4]

The most interesting is, of course, the behavior of the integrand $[\epsilon_k(x, t) - \epsilon_k(x, -t)]/t$ in the neighborhood of $t = 0$. Since the existence of $D_k f(x_0)$ implies the boundedness of f in the neighborhood of the point x_0, the assumption of integrability of f in Plessner's theorem is really no restriction of generality, and is made merely to simplify the statement.

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[2] See, for example, S. Kaczmarz, *Integrale vom Dinischen Typus*, Studia Mathematica vol. 3 (1931) pp. 189–199, or the author's *Trigonometric Series*, p. 77.

Kaczmarz shows that the set of continuous functions f for which the integral (1.4) is finite for *some* x is of the first category in the space of all continuous functions.

[3] The proof does not differ from that of the special case $r = 1$.

[4] Plessner, *Ueber das Verhalten anylytischer Funktionen auf dem Rande des Definitionsbereiches*, J. Reine Angew. Math. vol. 158 (1927) pp. 219–227. A different proof will be found in Marcinkiewicz, *Sur le séries de Fourier*, Fund. Math. vol. 27 (1936) pp. 38–69.

Recently, Marcinkiewicz proved the following remarkable result. Suppose that a function $f(x)$, of integrable square and of period 2π, has a finite derivative for every point x of a set E of positive measure. Then the integral

$$\int_0^\pi \frac{[f(x+t)+f(x-t)-2f(x)]^2}{t^3}\,dt \tag{1.7}$$

is finite for almost every $x \in E$.[5]

Consideration of this integral was suggested by its analogy with an important function

$$g(\theta)=\left\{\int_0^1 (1-\rho)\,|F'(\rho e^{i\theta})|^2 d\rho\right\}^{1/2} \qquad (F(z) \text{ regular in } |z|<1)$$

introduced by Littlewood and Paley.[6] However, the integral (1.7) may also be given a different interpretation. For, if we note that

$$\begin{aligned} f(x+t)+f(x-t)-2f(x) &= t[\omega_1(x,t)-\omega_1(x,-t)] \\ &= t[\epsilon_1(x,t)-\epsilon_1(x,-t)] \end{aligned}$$

at every point where $f'(x)$ exists, we may write the integral (1.7) in the form

$$\int_0^\pi \frac{[\omega_1(x,t)-\omega_1(x,-t)]^2}{t}\,dt=\int_0^\pi \frac{[\epsilon_1(x,t)-\epsilon_1(x,-t)]^2}{t}\,dt. \tag{1.8}$$

The finiteness of this integral at almost every point where f' exists, as compared with the fact that there exist continuous functions f such that

$$\int_0^\pi \frac{[\epsilon_0(x,t)-\epsilon_0(x,-t)]^2}{t}\,dt=+\infty \text{ for every } x$$

(cf. (1.5)), indicates that the behavior of the remainders $\epsilon_0(x,t)$ and $\epsilon_1(x,t)$ is essentially different. It also raises the problem of the extension of Marcinkiewicz's result to functions with a kth generalized derivative. The answer to that problem is given by the following theorem, the proof of which is the main object of this note, and which reduces to Marcinkiewicz's theorem for $k=1$.

THEOREM. *Suppose that a function $f(x)$ is of the class L^2 and of period 2π, and that the generalized derivative $D_k f(x)$ exists for every point x of*

[5] Marcinkiewicz, *Sur quelques intégrales du type de Dini*, Annales de la Société Polonaise de Mathématiques, 1938, pp. 42–50.

[6] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series*, Proc. London Math. Soc. vol. 42 (1937) pp. 52–89.

a set E of positive measure ($k=1, 2, \cdots$). Let $\omega_k(x, t)$ and $\epsilon_k(x, t)$ be defined by the equations

$$f(x+t) = \sum_{\nu=0}^{k-1} D_\nu f(x) t^\nu/\nu! + \omega_k(x, t) t^k/k!, \qquad x \in E.$$

$$\omega_k(x, t) = D_k f(x) + \epsilon_k(x, t),$$

Then the integral

$$(1.9) \qquad \int_0^\pi \frac{[\omega_k(x, t) - \omega_k(x, -t)]^2}{t}\, dt = \int_0^\pi \frac{[\epsilon_k(x, t) - \epsilon_k(x, -t)]^2}{t}\, dt$$

is finite for almost every $x \in E$.

This result complements the theorem of Plessner stated above, but at individual points is not comparable with the latter. It displays a new property of the remainder $\omega_k(x, t)$, a property which, unlike that in Plessner's theorem, is expressed by the convergence of a *positive* integral.

2. **Auxiliary lemmas.** The proof of the theorem depends on two known lemmas.[7]

LEMMA 1. *Suppose that P is a perfect set of positive measure situated in the interval $(0, 2\pi)$ and continued periodically. Let $\phi(x) = \phi_P(x)$ denote the function which is equal to 0 over P, and is equal to d if x belongs to an interval continguous to P and of length d. Then, whatever $\lambda > 0$, the integral*[8] $\int_{-\pi}^{\pi} (\phi^\lambda(x+t)/|t|^{\lambda+1})dt$ *is finite for almost every $x \in P$.*

LEMMA 2. *Suppose that a function $f(x)$ of period 2π has a kth generalized derivative for every point x of a set E of positive measure. Then $f(x) = g(x) + h(x)$, where*

(i) *$g(x)$ has everywhere a continuous kth derivative $g^{(k)}(x)$;*

(ii) *$g(x) = f(x)$ in a perfect set P contained in E and of positive measure;*

(iii) *except, perhaps, for a few segments Δ contiguous to P,*

$$\operatorname*{Max}_{x \in \Delta} |h(x)| \leqq M\Delta^k,$$

with M independent of Δ.[9]

[7] Both lemmas were used by Marcinkiewicz in his proof of the finiteness of the integral (1.7), so that the proof given here of the theorem is partly modelled on Marcinkiewicz's argument.

[8] J. Marcinkiewicz, the paper cited in footnote 4.

[9] J. Marcinkiewicz, loc. cit. Marcinkiewicz's argument shows that the set P may be of measure arbitrarily close to that of E.

Passing to the proof of the theorem, let us assume first that the function $f(x)$ has a continuous kth derivative $f^{(k)}(x)$. To fix the ideas we assume that k is even. Then

$$(2.1)\qquad \begin{aligned}(1/2)[f(x+t)-f(x-t)] &= f'(x)t + f'''(x)t^3/3! + \cdots \\ &\quad + f^{(k-1)}(x)t^{k-1}/(k-1)! \\ &\quad + (1/2)[\omega_k(x,t)-\omega_k(x,-t)]t^k/k!.\end{aligned}$$

Let us now consider the Fourier series of the function $f(x)$. It may be written in the form

$$f(x) \sim \sum_{\nu=-\infty}^{+\infty} c_\nu (i\nu)^{-k} e^{i\nu x},$$

if we assume, for simplicity, that the constant term of the Fourier series is zero. Of course,

$$\sum_{\nu=-\infty}^{+\infty} |c_\nu|^2 < +\infty,$$

and the Fourier series of $f^{(l)}(x)$, $0 \leqq l \leqq k$, is obtained by differentiating l times the Fourier series of $f(x)$. Hence, taking into account that for fixed t, $(1/2)[f(x+t)-f(x-t)] \sim \sum_{\nu=-\infty}^{+\infty} ic_\nu(i\nu)^{-k}e^{i\nu x} \sin \nu t$, we get (cf. (2.1))

$$\begin{aligned}&\frac{1}{2\cdot k!}[\omega_k(x,t)-\omega_k(x,-t)] \\ &\sim \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}\frac{1}{(i\nu t)^k}\left\{i\sin \nu t - \left(i\nu t + \frac{(i\nu t)^3}{3!} + \cdots + \frac{(i\nu t)^{k-1}}{(k-1)!}\right)\right\} \\ (2.2)\quad &= i(-1)^{k/2}\sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}\frac{1}{(\nu t)^k} \\ &\qquad \cdot\left\{\sin \nu t - \left[\nu t - \frac{(\nu t)^3}{3!} + \cdots \pm \frac{(\nu t)^{k-1}}{(k-1)!}\right]\right\} \\ &= i(-1)^{k/2}\sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}\xi_k(\nu t),\end{aligned}$$

say, where

$$\xi_k(u) = \left[\sin u - \left(u - \frac{u^3}{3!} + \cdots \pm \frac{u^{k-1}}{(k-1)!}\right)\right] \Big/ u^k.$$

Let us divide (2.2) by $t^{1/2}$ and apply Parseval's formula to the resulting relation. We get

$$\frac{1}{8\pi(k!)^2}\int_{-\pi}^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}dx=\sum_{\nu=-\infty}^{+\infty}|c_\nu|^2\frac{\xi_k^2(\nu t)}{t}.$$

Integrating both sides with respect to t over the interval $(0, \pi)$, and inverting the order of integration on the left, we see that

$$\frac{1}{8\pi(k!)^2}\int_{-\pi}^{\pi}dx\left\{\int_0^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}dt\right\} \tag{2.3}$$
$$=\sum_{\nu=-\infty}^{+\infty}|c_\nu|^2\int_0^{\pi}\frac{\xi_k^2(\nu t)}{t}dt=\sum_{\nu=-\infty}^{+\infty}|c_\nu|^2\int_0^{\nu\pi}\frac{\xi_k^2(u)}{u}du.$$

Let us now observe that the integral

$$\int_0^{\infty}\frac{\xi_k^2(u)}{u}du \tag{2.4}$$

is finite. This follows from the fact that in the neighborhood of $u=0$ the integrand is $(u^{-1})\cdot O(u^2)=O(u)=O(1)$, and in the neighborhood of $u=+\infty$ it is $u^{-1}\cdot O(u^{2k-2})u^{-2k}=O(u^{-2})$.

Since $\sum|c_\nu|^2<+\infty$, the right-hand side of (2.3) is finite, so that the integral

$$\int_0^{\pi}\frac{[\omega_k(x,t)-\omega_k(x,-t)]^2}{t}dt$$

is finite for almost every x.

If k is odd, the argument is similar. Instead of (2.1) we consider the formula

$$(1/2)[f(x+t)+f(x-t)]=f(x)+f''(x)t^2/2!+\cdots$$
$$+f^{(k-1)}(x)t^{k-1}/(k-1)!$$
$$+(1/2)[\omega_k(x,t)-\omega_k(x,-t)]t^k/k!$$

and we still get (2.3), where now

$$\xi_k(u)=[\cos u-(1-u^2/2!+\cdots\pm u^{k-1}/(k-1)!)]u^{-k}$$

and the integral (2.4) is again finite. Thus the theorem is proved if $f^{(k)}(x)$ exists everywhere and is continuous.

3. **Proof of the theorem.** In order to complete the proof of the theorem, let us assume that there is a periodic function $f\in L^2$ having a generalized derivative $D_kf(x)$ at every point of a set E of positive measure, and such that the integral (1.9) diverges everywhere in E. Let us now consider the decomposition $f=g+h$ of Lemma 2. The function g having everywhere a continuous derivative $g^{(k)}(x)$, the cor-

responding integral (1.9) is convergent almost everywhere, and so almost everywhere in E. If we could prove that the integral (1.9) corresponding to the function h is finite in a set E' contained in E and of positive measure, it would follow that the integral (1.9) for the function f is convergent almost everywhere in E', contrary to the assumption that it is divergent at every point of E. That contradiction would prove the theorem.

Let us therefore consider the function h of Lemma 2. That function vanishes over a perfect set $P \subset E$. Part (iii) of Lemma 2 asserts that for all, except perhaps a finite number, of the intervals Δ contiguous to P,

$$|h(x)| \leqq M\phi^k(x), \qquad x \in \Delta. \tag{3.1}$$

This inequality is also valid for the points of the set P, since both $h(x)$ and $\phi(x)$ vanish over P.

Let x_0 be any point of density of P (hence x_0 is not an end point of any interval contiguous to P). From the definition of the function $\phi(x)$ it follows that $\phi(x_0+t)=o(|t|)$ as $t\to 0$. Since (3.1) is valid in a neighborhood $(x_0-\delta, x_0+\delta)$ of x_0, we see that $h(x_0+t)=o(|t|^k)$ as $t\to 0$. Hence the function $\omega_k(x_0, t)$ corresponding to $h(x)$ is simply $k!h(x_0+t)t^{-k}$. Thus, in our case,

$$\begin{aligned}\int_0^\delta \frac{[\omega_k(x_0, t) - \omega_k(x_0, -t)]^2}{t}\,dt &\leqq 2\int_0^\delta \frac{\omega_k^2(x_0, t)}{t}\,dt + 2\int_0^\delta \frac{\omega_k^2(x_0, -t)}{t}\,dt \\ &= 2(k!)^2\int_{-\delta}^{+\delta} \frac{h^2(x_0+t)}{|t|^{2k+1}}\,dt \\ &\leqq 2M^2(k!)^2\int_{-\delta}^{+\delta} \frac{\phi^{2k}(x_0+t)}{|t|^{2k+1}}\,dt.\end{aligned}$$

If, therefore, E' is the set of points of density of P for which the integral

$$\int_{-\pi}^{\pi} \frac{\phi^{2k}(x+t)}{|t|^{2k+1}}\,dt$$

is finite, the integral (1.9) corresponding to the function h converges at every point of E'. This completes the proof of the theorem, since the set E' is obviously of positive measure.

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CONVERGENCE AND SUMMABILITY PROPERTIES OF SUBSEQUENCES

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In this paper we shall discuss the relation of the convergence or $(C, 1)$ summability of a sequence to that of its subsequences. Some analogous questions for subseries have been considered [6].[2]

Let $\{s_n\}$ be an arbitrary sequence. We can obtain a 1-1 map of its infinite subsequences on the interval $0<t\leqq 1$ as follows. Let $t=.a_1a_2a_3\cdots$ be the infinite dyadic expansion of a point t of the interval. Corresponding to this point we select the following subsequence: retain s_m if the mth place in the expansion, a_m, is 1, and drop it otherwise. The inverse correspondence is evident.

In terms of the Lebesgue measure of sets of points on (0, 1) we are now clearly in a position to speak of "almost all" or "almost none" of the subsequences of $\{s_n\}$.

The problem we have set ourselves is to determine under what conditions does the convergence or summability of a sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. For simplicity, we restrict ourselves to sequences of real numbers; although it is apparent that the results are more generally true.

In the case of convergence, the answer to our problem is simple: *a sequence is convergent if and only if almost all of its subsequences are convergent.* In the case of $(C, 1)$ summability, the problem is more difficult. It has been established that *all* the subsequences of a sequence cannot be summable by a fixed regular matrix method unless the sequence is in fact convergent [2]. In §§3, 4 we discuss, for $(C, 1)$ summability, the consequences of replacing *all* by *almost all.* We show, for example, that $\{s_n\}$ is $(C, 1)$ summable if almost all of its subsequences are, but not conversely.

The principal tools are the Rademacher functions, $R_n(t)$, and the properties of homogeneous sets.

Our results have certain obvious connections with probability; these are discussed in §5.

1. **Preliminary results.** All sets which occur in the sequel are to be taken as subsets of (0, 1); a.e. will mean "almost everywhere in (0, 1)."

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[2] Numbers in brackets refer to the references listed at the end of the paper.

We shall call a measurable set S homogeneous if it has the following property: let $t = .a_1a_2a_3 \cdots$ be the infinite dyadic expansion of a point t of S; then the point obtained by altering a finite number of the a_i also belongs to S. This definition is more restricted than the usual one [7], but is adequate for our purposes.

LEMMA 1. *A homogeneous set has measure* 0 *or* 1.

This is a well known result [7, p. 145]. The sets with which our theorems deal are homogeneous. In view of this lemma, the frequently occurring phrase "almost all the subsequences" can be replaced by the seemingly weaker but actually equivalent phrase "the subsequences corresponding to a set of positive measure."

LEMMA 2. *If S is of measure* 1, *then there exists a subset E, also of measure* 1, *with the property that if t belongs to E, so does $1-t$.*

This becomes immediately evident if the interval (0, 1/2) is folded symmetrically over onto the interval (1/2, 1). The device of symmetric points was used to advantage in [6].

LEMMA 3. *Let p_n be a sequence of positive numbers, increasing monotonically to infinity. If the series $\sum_{n=1}^{\infty} a_n/p_n$ converges, then* [5, p. 123]

$$\lim_{n\to\infty} \frac{1}{p_n} \sum_{1}^{n} a_k = 0.$$

For the properties of the Rademacher functions, we refer the reader to [3]. The principal one is contained in the following lemma.

LEMMA 4. *The series $\sum_1^{\infty} a_k R_k(t)$ converges on a set of measure* 1 *or measure* 0, *according as $\sum_1^{\infty} a_k^2$ converges or diverges.*

LEMMA 5. *If $\sum_1^{\infty} s_k^2/k^2$ converges, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} s_k R_k(t) = 0 \text{ a.e.} \tag{1.1}$$

For the hypothesis implies, by Lemma 4, that $\sum_1^{\infty} (s_k^2/k^2) R_k(t)$ converges a.e., so that the result follows from Lemma 3, with $p_n = n$.

LEMMA 6. *If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} s_k R_k(t) = L(t) \tag{1.1'}$$

exists a.e., then $L(t) = 0$ a.e. (so that (1.1) *is true) and*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_1^n s_k^2 = 0. \tag{1.2}$$

(1.1′) must hold uniformly on a set A of positive measure. Hence, for some M,

$$|L(t)| \leqq M, \qquad t \in A.$$

But the set of points for which this inequality holds is homogeneous, so that, by Lemma 1, $L(t)$ is bounded a.e. Since it is obviously integrable,

$$\int_0^1 L(t)dt = a \tag{1.3}$$

exists.

Now, choose an $\epsilon > 0$ and consider the sets I_1, I_2 and I_3 for which

$$L(t) - a > \epsilon,$$
$$L(t) - a < -\epsilon,$$
$$|L(t) - a| \leqq \epsilon,$$

respectively. These sets are homogeneous; since their union is of measure 1, at least one of them must be of measure 1. If either I_1 or I_2 has measure 1, (1.3) is contradicted. Hence $|L(t)-a| \leqq \epsilon$ a.e., so that $L(t)=a$ on a set B of measure 1.

By Lemma 2, we can pick two points t_0 and $1-t_0$ in B. By (1.1′)

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n s_k R_k(t_0) = a,$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_1^n s_k R_k(1 - t_0) = a.$$

Adding these, we see that $a=0$; this establishes (1.1).

To prove (1.2) we employ a device due to Kolmogoroff [4, p. 127]. Let $S_{m,n}(t) = \sum_m^n s_k R_k(t)$. Then

$$\begin{aligned} S_{m,n}^2(t) &= \sum_{j,k=m}^n s_j s_k R_j(t) R_k(t) \\ &= \sum_m^n s_k^2 + 2 \sum_{m \leqq j < k \leqq n} s_j s_k R_j(t) R_k(t). \end{aligned} \tag{1.4}$$

If (1.1) holds for almost all t, it must hold uniformly on a set E of positive measure $|E|$. By (1.4)

$$\int_E S_{m,n}^2(t)dt = |E| \sum_m^n s_k^2 + M \tag{1.5}$$

where

$$M = 2 \sum_{m \leq j < k \leq n} s_j s_k \int_E R_j(t) R_k(t) dt.$$

From the Schwarz inequality

$$|M| \leq 2\left(\sum_{m \leq j < k \leq n} s_j^2 s_k^2 \right)^{1/2} \left(\sum_{m \leq j < k \leq n} b_{jk}^2 \right)^{1/2} \tag{1.6}$$

where $b_{jk} = \int_E R_j(t) R_k(t) dt$.

The functions $R_j(t)R_k(t)$, for $1 \leq j < k < \infty$, are orthonormal on $(0, 1)$. If $X(t)$ is the characteristic function of E, it follows from Bessel's inequality that

$$\sum_{1 \leq j < k < \infty} b_{jk}^2 \leq \int_0^1 [X(t)]^2 dt = |E|.$$

For a large enough value of m

$$\left(\sum_{m \leq j < k < \infty} b_{jk}^2 \right)^{1/2} \leq |E|/4.$$

By (1.6)

$$|M| \leq \left(\sum_{j,k=m}^n s_j^2 s_k^2 \right)^{1/2} (|E|/2) \leq (|E|/2) \sum_m^n s_j^2.$$

Then, by (1.5) we have

$$\int_E S_{m,n}^2(t)dt \geq (|E|/2) \sum_m^n s_j^2. \tag{1.7}$$

But for this fixed value of m,

$$\lim_{n \to \infty} \frac{1}{n^2} S_{m,n}^2(t) = 0$$

boundedly on E. (1.7) then implies that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_m^n s_j^2 = 0$$

and the proof is complete.

2. **Convergence of subsequences.** If a sequence $\{s_n\}$ of real numbers is convergent to S, then every subsequence is likewise convergent to S. What may be said if $\{s_n\}$ is divergent?

THEOREM 1. *If $\{s_n\}$ is divergent, so are almost all of its subsequences.*

Let C be the set of points corresponding to convergent subsequences. It is homogeneous, and hence of measure 0 or 1. Suppose it is of measure 1. By Lemma 2, we can pick a number t_0 not of the form $K/2^n$, such that both t_0 and $1-t_0$ belong to C. The corresponding subsequences can have no terms in common, for the dyadic expansions of t_0 and $1-t_0$ cannot agree anywhere. Furthermore, every term of $\{s_n\}$ is present in one or the other of the subsequences. Thus we have split $\{s_n\}$ into two subsequences, and since $t_0 \in C$, $1-t_0 \in C$, each is convergent. Let their limit points be S' and S''.

Every convergent subsequence of $\{s_n\}$ must then have either S' or S'' as its limit. Let C' (C'') be the set of points corresponding to subsequences convergent to S' (S''). Then C' and C'' are homogeneous and, since $C = C' \cup C''$, one of them—say C'—must be of measure 1. This means that the t_0 above could have been chosen from C', so that $\{s_n\}$ can be split into two convergent subsequences, each of which converges to the *same* limit. Every subsequence is then convergent, and hence $\{s_n\}$ is itself convergent; this contradicts the hypothesis of our theorem, and C therefore has measure 0.

COROLLARY. *If the series $\sum_1^\infty a_k$ converges for almost every bracketing of terms, it is convergent.*

For if $\{s_n\}$ is the sequence of partial sums, and we bracket the series in blocks of length $n_1, n_2, \cdots$, then the partial sums of the resulting series are $s_{n_1}, s_{n_1+n_2}, \cdots$, a subsequence of $\{s_n\}$. By the theorem, if almost all of these converge, the sequence itself does.

3. **$(C, 1)$ summability.** In this and the succeeding section, we consider the corresponding problem for $(C, 1)$ summability. As we have previously noted, we cannot expect all of the subsequences of a sequence to be $(C, 1)$ summable unless the sequence is convergent.

THEOREM 2. *If $\{s_n\}$ is $(C, 1)$ summable to S, and*

$$\sum_1^\infty s_k^2/k^2 < \infty, \tag{3.1}$$

then almost all the subsequences are $(C, 1)$ summable to S.

The problem of the convergence of the Cesàro means of almost all the subsequences of $\{s_n\}$ reduces to the requirement that

$$\lim_{n\to\infty} \frac{\sum_1^n s_k \cdot \frac{1}{2}[1 + R_k(t)]}{\sum_1^n \frac{1}{2}[1 + R_k(t)]} = S \text{ a.e.} \tag{3.2}$$

We can rewrite the expression inside the limit as

$$\frac{\frac{1}{n}\sum_1^n s_k + \frac{1}{n}\sum_1^n s_k R_k(t)}{1 + \frac{1}{n}\sum_1^n R_k(t)}. \tag{3.3}$$

By hypothesis, $\lim (1/n)\sum_1^n s_k = S$; then (3.2) follows immediately by condition (3.1) and Lemma 4.

The question arises whether the condition (3.1) can be removed or at least weakened. We shall show in the next section that it cannot be dropped entirely; on the other hand, this does not preclude the possibility that it might be weakened, perhaps to (1.2) which by Lemma 3 it implies, and which is *necessary* for the validity of the conclusion (Theorem 3, corollary).

4. **Converse.** We now prove the analogue of Theorem 1 for $(C, 1)$ summability. We note again that since $(C, 1)$ summability is a method that preserves translation, our sets are homogeneous.

THEOREM 3. *If almost all the subsequences of $\{s_n\}$ are $(C, 1)$ summable, then $\{s_n\}$ is itself summable to a value S, and almost all the subsequences are in turn summable to S.*

As in the preceding section, the hypothesis implies that (3.3) converges a.e. as $n\to\infty$. By Lemma 5, the denominator of (3.3) approaches the value 1 a.e., so that

$$\sigma_n(t) = \frac{1}{n}\sum_1^n s_k[1 + R_k(t)] \tag{4.1}$$

converges on a set C of measure 1. By Lemma 2, we can pick t_0 and $1 - t_0$ belonging to C; then

$$\sigma_n(t_0) + \sigma_n(1 - t_0) = \frac{2}{n}\sum_1^n s_k$$

so that $(1/n)\sum_1^n s_k$ converges to a value S. From (4.1) we see that

$$\frac{1}{n}\sum_1^n s_k R_k(t) \tag{4.2}$$

converges a.e.; by Lemma 6, this last expression then converges a.e. to 0, and, by (4.1),

$$\lim \sigma_n(t) = S \text{ a.e.}$$

COROLLARY. *Under the hypothesis of the theorem,*

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_1^n s_k^2 = 0. \tag{4.3}$$

For (4.2) converges a.e.; (4.3) follows from Lemma 6.

We are now in a position to answer the question raised in the preceding section by exhibiting a $(C, 1)$ summable sequence with almost none of its subsequences $(C, 1)$ summable. This example is the sequence $s_n=(-1)^n n^{1/2}$. It is readily seen to be summable to zero, but since it violates (4.3), almost none of its subsequences can be $(C, 1)$ summable.

If the sequence $\{s_n\}$ is bounded, condition (3.1) is satisfied. Combining Theorems 2 and 3, we have the following theorem.

THEOREM 4. *A bounded sequence is $(C, 1)$ summable if and only if almost all of its subsequences are $(C, 1)$ summable.*

5. **Probability.** Lemma 5, on which a large part of our paper depends, can also be obtained from the strong law of large numbers [4, p. 59], for we may regard the sequence $s_nR_n(t)$ as a sequence of independent random variables of mean value 0 and standard deviation s_n^2.

An interesting result that may be obtained as a corollary of Theorem 2 is one of Birnbaum and Zuckerman on the v. Mises collective [1].

THEOREM A. *If a sequence of 0's and 1's satisfies the first postulate of v. Mises, the second postulate is fulfilled for almost all selections.*

Here the v. Mises postulates are:

P1: *If n_0 and n_1 are the number of 0's and 1's respectively among the first n terms of the sequence, then the limits* $\lim n_0/n$ *and* $\lim n_1/n$ *exist.*

P2: *For a suitable selection of an infinite subsequence from the given sequence, the same limits exist and their values are unchanged.*

To obtain the theorem we need only observe that $\lim n_1/n$ is the

(*C*, 1) sum of the given sequence, and that (3.1) is automatically satisfied.

References

1. Birnbaum and Zuckerman, *On the properties of a collective*, Amer. J. Math. vol. 62 (1940) pp. 787–791.

2. R. Creighton Buck, *A note on subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 898–899.

3. Kacmarz and Steinhaus, *Theorie der orthogonalreihen*, Warsaw, 1935, pp. 125–132.

4. A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichskeitsrechnung*, Ergebnisse der Mathematik und ihrer grenzgebiete, vol. 2, no. 3, Berlin, 1933.

5. K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, Berlin, 1922.

6. H. Pollard, *Subseries of a convergent series*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 730–731.

7. C. Visser, *The law of nought-or-one*, Studia Mathematica vol. 7 (1938) pp. 143–159.

Harvard University

LAMBERT SUMMABILITY OF ORTHOGONAL SERIES

RICHARD BELLMAN

If we define Lambert summability of a series, $\sum_1^\infty a_n$, in terms of the existence of the limit

$$L(a_n) = \lim_{x \to 1-0} (1-x) \sum_1^\infty \frac{n a_n x^n}{1-x^n} \tag{1}$$

we have, by a well known theorem of Hardy-Littlewood [1],[1] that $C(a_n) \to L(a_n) \to A(a_n)$; $C(a_n)$, $A(a_n)$ are respectively the Cesàro and Abel means of the series $\sum_1^\infty a_n$.

The proof of $C(a_n) \to L(a_n)$ is elementary in nature, but the proof of $L(a_n) \to A(a_n)$ requires the prime number theorem, and conversely the theorem $L(a_n) \to A(a_n)$ implies the prime number theorem.

For that reason, it is perhaps interesting to show that for orthogonal series of functions $f(x)$, belonging to L^2, the inclusion of $L(a_n)$ between $C(a_n)$ and $A(a_n)$ follows in completely elementary fashion.

That $C(a_n) \sim A(a_n)$ for orthogonal series of L^2 is a known result of Kaczmarz [2]. Hence it is sufficient to show that $L(a_n) \to C(a_n)$. In addition, it is further known that $C(a_n)$ is equivalent to the convergence of the partial sums of the orthogonal series $s_{2^n}(\theta) = \sum_1^{2^n} a_k \phi_k(\theta)$ [3]. Therefore, finally, it comes to showing that Lambert summability implies the convergence of the partial sums $s_{2^n}(\theta)$, in order to prove the theorem.

Let $f(\theta) \subset L^2(a, b)$, $a_n = \int_a^b f(\theta)\phi_n(\theta)d\theta$; where $(\phi_n(\theta))$ is an orthonormal sequence in (a, b), $s_n(\theta) = \sum_1^n a_n \phi_n(\theta)$.

Write, where x is $1 - 1/2^n$,

$$U_n(\theta) = \sum_1^\infty k a_k \phi_k(\theta) \frac{(1-x)x^k}{1-x^k} - s_{2^n}(\theta) = T_n(\theta) + V_n(\theta) \tag{2}$$

where

$$T_n(\theta) = \sum_1^{2^n} a_k \phi_k(\theta) \left(\frac{k(1-x)x^k}{1-x^k} - 1 \right), \tag{3}$$

$$V_n(\theta) = \sum_{2^n+1}^\infty k a_k \phi_k(\theta) \frac{(1-x)x^k}{1-x^k}. \tag{4}$$

If $\lim_{n \to \infty} U_n(\theta) = 0$, the result is proven. To that end, consider the

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[1] Numbers in brackets refer to the references listed at the end of the paper.

series

$$\sum_1^\infty [U_n(\theta)]^2. \tag{5}$$

To prove convergence almost everywhere in θ, it is sufficient to show

$$\sum_{n=1}^\infty \int_a^b [U_n(\theta)]^2 d\theta < \infty. \tag{6}$$

We have

$$\sum_n \int_a^b [U_n(\theta)]^2 d\theta \leqq 2 \sum_n \int_a^b [T_n(\theta)]^2 d\theta + 2 \sum_n \int_a^b [V_n(\theta)]^2 d\theta. \tag{7}$$

Let us consider the convergence of each series separately.

$$\begin{aligned} \sum_n \int_a^b [T_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_1^{2^n} a_k \phi_k(\theta) \left(\frac{k(1-x)x^k}{1-x^k} - 1 \right) \right)^2 d\theta \\ &= \sum_n \left(\sum_1^{2^n} a_k^2 \left(\frac{k(1-x)x^k}{1-x^k} - 1 \right)^2 \right) \end{aligned} \tag{8}$$

where the x appearing in $\sum_1^{2^n}$ is $1-1/2^n$, $n \geqq 1$.

Now

$$1 - x^k \leqq k(1-x), \qquad 0 \leqq x \leqq 1, \tag{9}$$

$$1 - x^k \geqq 1 - \frac{k(1-x)x^k}{1-x^k} \geqq 0, \tag{10}$$

so that

$$\begin{aligned} \sum_n \int_a^b [T_n(\theta)]^2 d\theta &\leqq \sum_n \sum_1^{2^n} a_k^2 (1-x^k)^2 \leqq \sum_n \sum_1^{2^n} k^2 a_k^2 (1-x)^2 \\ &\leqq \sum_n \frac{1}{2^{2n}} \sum_1^{2^n} k^2 a_k^2 \leqq \sum_k a_k^2 k^2 \sum_{n \geqq \log_2 k} 2^{-2n} \leqq A \sum_k a_k^2 \end{aligned} \tag{11}$$

and $\sum_k a_k^2 < \infty$ since $f(x) \subset L^2(a, b)$.

Now for the second series $\sum_n \int_a^b [V_n(\theta)]^2 d\theta$:

$$\begin{aligned} \sum_n \int_a^b [V_n(\theta)]^2 d\theta &= \sum_n \int_a^b \left(\sum_{2^n+1}^\infty a_k \phi_k(\theta) \frac{k(1-x)x^k}{1-x^k} \right)^2 d\theta \\ &= \sum_n \left\{ \sum_{2^n+1}^\infty k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2} \right\} \end{aligned} \tag{12}$$

where the x appearing in $\sum_{2^n+1}^{\infty}$ is $1-1/2^n$, $n \geqq 1$.

Since $(1-2^{-n})^k$ is a decreasing function of k,

$$\sum_n \sum_{2^n+1}^{\infty} k^2 a_k^2 \frac{(1-x)^2 x^{2k}}{(1-x^k)^2}$$

$$\leqq \sum_n \frac{1}{1-(1-2^{-n})^{2^n}} \sum_{2^n+1}^{\infty} k^2 a_k^2 (1-x)^2 x^{2k} \tag{13}$$

$$\leqq A \sum_n \sum_{2^n+1}^{\infty} k^2 a_k^2 (1-x)^2 x^{2k}.$$

We can majorize $k^2 \sum_1^{\infty} 2^{-2n}(1-2^{-n})^{2k}$ by the integral

$$k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x-1})^{2k} dx = 4k^2 \int_1^{\infty} 2^{-2x}(1-2^{-x})^k dx$$

$$< 4k^2 \int_0^{\infty} 2^{-2x}(1-2^{-x})^{2k} dx \tag{14}$$

$$= A k^2/(2k+1)(2k+2)$$

which is obviously bounded.

Therefore we have proven the convergence of the series, which implies that $\lim_{n\to\infty} U_n(\theta) = 0$ almost everywhere in θ, which implies that

$$L(a_n) = \lim_{n\to\infty} s_{2^n}(\theta) \tag{15}$$

almost everywhere in θ.

This is equivalent to what we set out to prove.

References

1. Hardy and Littlewood, *On a Tauberian theorem for Lambert's series*, Proc. Lond. Math. Soc. (2) vol. 19 (1921) pp. 21–29.

2. Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, Warsaw, 1935, Theorem 583.

3. Ibid., Theorem 585.

University of Wisconsin

A NOTE ON PRIMITIVE SKEW CURVES

DICK WICK HALL

In 1930 Kuratowski[1] established the following result:

THEOREM A. *A locally connected continuum, containing but a finite number of simple closed curves, is homeomorphic with a subset of the plane, provided that it does not contain a primitive skew curve of type* I *or a primitive skew curve of type* II.

By a primitive skew curve of type I we mean any topological image of the complex C which consists of two groups of three vertices each and nine 1-cells, in a fashion that each vertex of one group together with each vertex of the other group bounds a 1-cell. By a primitive skew curve of type II we mean any topological image of the complex D which consists of five vertices and ten 1-cells in a fashion that each pair of vertices bounds a 1-cell.

In 1934 Claytor[2] proved that every cyclic locally connected continuum containing no primitive skew curve of either type must be homeomorphic with a subset of a spherical surface.

In this note we point out that for a large class of locally connected continua the property of being planar may be insured merely by requiring that the given locally connected continuum contain no primitive skew curve of type I. Stated precisely, our principal theorem is the following:

THEOREM 1. *Let M be a locally connected continuum separated by no pair of its points and assume that M contains no primitive skew curve of type* I. *Then M contains no primitive skew curve of type* II.

The proof is immediate. Assuming the existence of a primitive skew curve D of type II in M and using the fact that no two points separate M we may easily find a simple arc ab in M having only the points a and b in common with D and such that a is interior to a free arc of D while b does not lie on the closure of this free arc. Denote the sum of D and the arc ab by D'. A careful examination of D' shows that this set must contain a primitive skew curve of type I contrary to our hypothesis on M.

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[1] *Sur les problème des courbes gauches en topologie*, Fund. Math. vol. 15 (1930) pp. 271–283.

[2] *Topological immersion of Peanian continua in a spherical surface*, Ann. of Math. vol. 35 (1934) pp. 809–835.

It is natural to inquire whether or not the above result remains true when the types of primitive skew curves there mentioned are interchanged. To show that this is not the case we need only construct a locally connected continuum K which is separated by no pair of its points, but such that K contains a primitive skew curve of type I but no primitive skew curve of type II. To do this let K consist of three 2-cells E, F, G having their interiors disjoint and exactly three boundary points a, b, c in common. It is self-evident that no two points separate K while K contains a primitive skew curve of type I. It is easy to show that K contains no primitive skew curve of type II by making use of the fact that no two vertices of such a primitive skew curve can be separared by the removal of any three points of the curve.

As an application of our theorem we prove the following unpublished theorem of F. B. Jones:[3]

THEOREM 2 (JONES). *Let M be a locally connected continuum separated by no pair of its points but by every one of its simple closed curves, and suppose that M contains no primitive skew curve of type* I. *Then M is homeomorphic with a spherical surface.*

PROOF. By Theorem 1 and the theorem of Claytor cited above we know that M is homeomorphic to a subset of a spherical surface S. If the theorem is false then M has a complementary domain D in S. By a theorem of R. L. Moore[4] the boundary of D is a simple closed curve J. Now J must separate M. Using this fact together with the knowledge that every component of $M-J$ must lie on the side of J opposite to D we easily see that some pair of points of M must separate M. This contradiction completes the proof of the theorem.

UNIVERSITY OF MARYLAND

[3] See Bull. Amer. Math. Soc. abstract 48-11-340.

[4] *Concerning the common boundary of two domains*, Fund. Math. vol. 6 (1924) pp. 203–213.

NOTE ON THE NON-EXISTENCE OF ODD PERFECT NUMBERS OF FORM $p^{\alpha}q_1^2q_2^2\cdots q_{t-1}^2q_t^4$

ALFRED BRAUER

A paper under this title appears in Bull. Amer. Math. Soc. vol. 49 (1943) pp. 712–718. On October 2, 1943, I received a photoprint of a paper of H. J. Kanold, *Verschärfung einer notwendigen Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, J. Reine Angew. Math. vol. 184 (1942) pp. 116–124, for reviewing in the Mathematical Reviews. This paper was not available to me previously. In this paper the author, too, proves my theorem. But his proof is so much more difficult than my proof that I believe that the publication of my paper is also now not superfluous. The new method of both papers is the proof of case I. While my proof consists of only 8 lines, Kanold needs almost 5 pages for the proof of this case. In the proof of II, well known methods can be applied, and therefore it is not surprising that both proofs are similar. Difficulties arise here only if we obtain divisors which are greater than the range of D. N. Lehmer's prime factor tables.

Thus in the proof of II c, I had to consider $\sigma(30941^2)$. I checked the small primes and found that every prime factor must be greater than 151. Using this fact, and Lemma 1 and 2, I could prove this case without knowing the factorization of $\sigma(30941^2)$. I learned from Kanold's paper that just the next prime, namely 157, is a divisor,

$$\sigma(30941^2) = 957{,}376{,}423 = 157\cdot 433\cdot 14083.$$

Now II c can be proved in a simpler, but perhaps less interesting way. Since $s=13$ and $k=2$ one of the three factors of $\sigma(30941^2)$ equals p and the others must be q_1 and q_2. But then $(p+1)/2$ and hence n must be divisible by 79 or by 7. This contradicts $k=2$.

Moreover, I learned from Kanold's paper that he proved the special cases $\alpha=1$ and $\alpha=5$ already in an earlier paper *Über eine notwendige Bedingung für die Existenz einer ungeraden vollkommenen Zahl*, Deutsche Mathematik vol. 4 (1939) pp. 53–57, not available here and not mentioned in his paper cited in footnote 5.

UNIVERSITY OF NORTH CAROLINA

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A CORRECTION TO "ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION"[1]

M. KAC

Dr. R. P. Boas has kindly called my attention to the fact that the proofs of the results of Littlewood and Offord[2] have been published in the Proc. Cambridge Philos. Soc. vol. 35 (1939) pp. 133–148. It should also be mentioned that a similar problem was studied by entirely different methods by S. O. Rice in his paper *The distribution of the maxima of a random curve*, Amer. J. Math. vol. 61 (1939) pp. 409–416. Finally, I take this opportunity to state that under the conditions considered in my note the probability that equation (1) has a multiple root is zero. In fact, this probability is the probability that the discriminant vanish; this probability is clearly equal to zero since the discriminant is an analytic function of the coefficients and the measure considered is nonsingular.

CORNELL UNIVERSITY

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[1] Bull. Amer. Math. Soc. vol. 49 (1943) pp. 314–320.

[2] See the footnote on page 314 of my note.

A CORRECTION TO "A LINEAR TRANSFORMATION WHOSE VARIABLES AND COEFFICIENTS ARE SETS OF POINTS"

S. T. SANDERS, JR.

In my paper *A linear transformation whose variables and coefficients are sets of points*, published in Bull. Amer. Math. Soc. vol. 48 (1942), the fourth line, page 442, should read: "there is a subscript $k=k'$ such that $P_{k'}$ involves only factors of P_1."

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ERRATA TO "TABLE OF THE ZEROS OF THE LEGENDRE POLYNOMIALS OF ORDER 1-16 AND THE WEIGHT COEFFICIENTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA"[1]

ARNOLD N. LOWAN, NORMAN DAVIDS, AND ARTHUR LEVENSON

The following errors should be noted.

Page 740. The statement "where ξ is a point in the interval (p, q) and k_n is the normalizing factor for $P_n(x)$, which is equal to

$$\left(\frac{2n+1}{q-p}\right)^{1/2}."$$

should be replaced by "where ξ is a point in the interval (p, q) and k_n is

$$({}_{2n}C_n)(2n+1)^{1/2}/(q-p)^{n+1/2}."$$

The bottom of the same page should contain the following:

"*Note.* On this and the previous page, the subscripts of x_n and a_n are allowed to range from 1 to n. In the table, however, it seemed more convenient to change the notation slightly, and permit both positive and negative subscripts. Thus for $n=3$, the roots are x_{-1}, x_0, x_1. Roots with negative subscripts can be obtained by symmetry from those tabulated. Similarly for the entries a_n."

Page 741. The value of x_2 should read 0.519096129206812.

Page 742. The value of x_1 should read $x_1=0.125233408511469$.

The value of x_2 should read $x_2=0.367831498998180$.

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E. J. Gumbel, *On the plotting of statistical observations*, abstract 49-9-238.

p. 698, line 5 of the abstract. Instead of "The correction Δ for the rank is unlimited and possesses a mode, Δ increases" read "If the variate is unlimited and possesses a single mode the correction Δ for the rank increases."

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A MANUAL FOR AUTHORS OF MATHEMATICAL PAPERS*

The purpose of this manual is to help mathematicians in preparing papers for publication. A reasonable amount of attention on the part of authors to the rules and suggestions given below will save the time of editors, referees, printers, and readers, and will help keep the cost of publication at a minimum. Although most of the discussion will apply to mathematical papers in general, special rules will occasionally be given for papers specifically intended for the Bulletin or Transactions of the American Mathematical Society.

Style. In the course of writing a mathematical paper intended for publication, the author must settle such questions of usage and arrangement as (1) the organization of the paper into chapters, sections, paragraphs, propositions, theorems, proofs, examples, remarks, and the use of appropriate headings; (2) the adoption of a system of enumeration for the sections, theorems; (3) the treatment of footnotes, references and bibliography; (4) the use of capital or small initial letters in such adjectives as abelian, euclidean, jacobian; (5) the use or non-use of hyphens in such technical terms as sub-harmonic, fixed-point; (6) the questions of style which arise when words in the text are replaced by symbols.

Concerning the first three questions—particularly the third—the author's decisions must to some extent be governed by the routine stylistic requirements of the journal for which his paper is intended. For guidance, the author should consult current numbers of that journal; if his paper is intended for the Bulletin or Transactions, he may consult, instead, the Special Rules given below.

Concerning the remaining questions, perhaps the only general rule that can be offered is the Rule of Uniformity: Whatever the usage adopted by an author in a given paper, *that usage should be maintained consistently throughout the paper*. It may be remarked that the reasons for urging a careful observance of this rule are economic as well as stylistic. Every paper is carefully checked by an editorial assistant before it is sent to the printer. An author who has written "euclidean" part of the time and "Euclidean" the rest of the time may have done so inadvertently. But the assistant cannot guess this.

* The writers of this manual have received invaluable advice from members of the office staff of the American Mathematical Society and have consulted freely the following excellent pamphlets: *Notes on the preparation of mathematical papers* (The London Mathematical Society); *Typographic suggestions to authors* (George Banta Publishing Co.); *Author's manual* (Duke Mathematical Journal).

Nor can one guess whether the author prefers the capital or the small letters. The correspondence between author and editorial staff which must take place in order to settle the matter causes delay and expense which could easily have been avoided.

The use of mathematical symbols in the solid text is not only permissible but is frequently encouraged by editors as a means for securing greater economy of space on the printed page. Authors, however, should consider carefully the questions of style which arise from this practice, particularly in the use of "$<$" and "$=$". Such phrases as "assume $x>y$" and "assume that x is $>y$" are in common use and there appears to be no widely accepted ground for preferring one to the other. On the other hand, no one would write "assume that x and y are $=$" although the objection appears to be largely visual. Some writers do not hesitate to begin a sentence with a mathematical symbol, but others regard this as a barbarism.[1] In the complete absence of agreement on these questions, we offer only the following general rules in addition to the Rule of Uniformity:

1. Symbols in the text should be used in such a way that their translation into words or phrases will require no effort on the part of the reader.

2. The resulting translation should conform to the standards of good literary composition.

Rules 1 and 2 are admittedly not decisive in all cases. Thus "let $\epsilon \geqq 0$ be a number such that $\cdots$" would be objectionable on both counts, but is probably preferable, in appearance at least, to "let ϵ be a number $\geqq 0$ such that $\cdots$." One could, of course, avoid both forms by writing "let ϵ be a non-negative real number $\cdots$." Again, "$\cdots$ the expression on the left is $<$ the smallest value of $\cdots$" satisfies rules 1 and 2. Yet the substitution of two words for "$<$" would result in a far better appearance. This improvement is surely worth the slight extra effort involved.

The printing of mathematical symbols. There are unfortunately many useful arrangements of symbols which must be avoided in a manuscript intended for publication, on account of the difficulties which would have to be overcome by the compositor. A type consists of a solid block or "body" with a raised character on one of its rectangular faces. The bodies are normally laid side by side by a machine and form a row of uniform width. Besides the ordinary

[1] There is a similar division of opinion concerning the use of the "hanging theorem" in which the word THEOREM is simultaneously the first word of a paragraph and the last word of the sentence which precedes it.

alphabets, most printers have on hand a number of "dashed letters" ($\overline{A}$, $\bar{a}$, $\cdots$) and "tilde letters" ($\tilde{A}$, $\cdots$). If a dashed letter, say $\bar{a}$, is not in the printer's possession, it does not follow that this letter cannot be printed. But the typesetting involved is much more complicated and includes a number of operations by hand. If $\bar{a}$ occurs in the solid text, the body of an ordinary a must be reduced in size by filing, to make room for a thin body carrying the bar. The two bodies must then be placed in position by hand. The thin body may or may not stay in place and thus the alignment of the bar may or may not be correct. Moreover, the bars, when printed, may not always be uniform. Thus to the extra expense involved in the formation and typesetting of the symbol $\bar{a}$, it is necessary to add the cost of re-alignment or re-setting in case of mishaps during the printing of proof.

The typesetting of inferior and superior symbols (subscripts and superscripts) in the solid text offers no difficulty when those symbols are available as *inferiors and superiors* (see Appendix C) and when no symbol is to be placed directly above or below another. Thus, $(a+bx)^{n/3}$, A^2 can be set by machine, but $x^{\bar{3}}$, A_3^2 require hand work.[2] Subscripts to superscripts and superscripts to subscripts, all belonging to a single term, are extremely difficult to set, even by hand.

In displayed formulas, the insertion of bars, subscripts and the like is somewhat less difficult since there is space above and below, whereas in the solid text, the compositor has only a single line with which to work. Thus the symbol A_3^2 no longer requires hand work since the superscript 2 can be set by machine in the subscript position of the line above the line of the A. Note however that when set in this manner, the superscript will be slightly higher: A^2_3 rather than A_3^2.

Special rules. Authors of papers intended for the Bulletin or Transactions should give special attention to the following rules. A reasonably strict observance of these rules will help keep printing costs down, prevent an uneconomical use of space, insure a certain uniformity of appearance, and will greatly simplify the task of final preparation of manuscripts for the printer.

1. Use only such characters as the printer has in stock. The list in Appendix C includes all special types now available for the Bulletin and Transactions. Additions to this list are made from time to time. Note particularly what letters and numbers are available in sub-superscript (inferior to superior) and super-subscript positions.

[2] These remarks are not intended to imply that symbols such as A_3^2 must never be used in the solid text, rather that the author should exercise a measure of discretion.

Note also that many symbols are not available in footnote size. For this reason, if for no other, complicated formulas should not be put into the footnotes. In this connection it might be remarked that an excessive number of footnotes sometimes gives the undesirable impression that the paper is being "written in the footnotes." Careful organization, however, always makes it possible to reduce the number and length of footnotes, even to eliminate them entirely.

2. Do not put into the solid text any formula which would require spreading (uneven spacing) of lines.

3. In the solid text, write $\sum_{i=1}^{n}$; $\prod_{i=1}^{n}$; $\lim_{n\to\infty}$; $\int_a^b$. In the first two examples the n should be directly over the i. In the integral the b should be directly over the a; it is not feasible to place the a nearer the integral sign since the latter is on a rectangular face: $\boxed{\int}$. In displayed formulas, the limits should occupy only one line. For example, write

$$\sum_{i=i_0,\, j=j_0}^{\infty} \qquad \text{instead of} \qquad \sum_{\substack{i=i_0 \\ j=j_0}}^{\infty}$$

4. In the text, replace $e^{(\ )}$ by exp ( ) if the expression in the parentheses is complicated.

5. Use $C_{m,n}$ (binomial coefficient) instead of $\binom{m}{n}$.

6. In the text, write $\sim_a$, in display $\sim$.

7. Use $\in$ to mean "is a member of," ϵ for epsilon. The symbol for "is not a member of" is $\notin$.

8. Use fractional exponents in place of radicals. A fraction occurring as subscript or superscript should be formed with a solidus, for example 2/3. In display, the number of lines occupied by a fraction should be as small as possible. Frequently the number of lines can be reduced by a simple rearrangement. For example:

$$\frac{\cos \dfrac{1}{x}}{\sqrt{a + \dfrac{b}{x}}} \qquad \text{can be written} \qquad \frac{\cos\,(1/x)}{(a + b/x)^{1/2}}$$

9. Long formulas, or expressions which might break awkwardly at the end of a line of text, should be displayed.

10. In numbering sections, theorems, displayed formulas, and so on, the author should adopt a system which is simple and self-explanatory. A system based on such symbols as II.3.6.1a is unnecessarily complicated.

11. Footnotes should be assembled at the end of the paper, typed double spaced, and numbered with arabic numbers. The footnote symbols currently in use are illustrated as follows: "It was proved[2] that · · · " (Bulletin); "It was proved (2) that · · · " (Transactions). A footnote symbol should not follow a mathematical symbol. Footnotes should be numbered in the order in which they appear, beginning with 1 and continuing throughout the paper. Do not use any symbol at all for the footnote "Presented to · · · " which frequently appears on the title page.

12. Mathematical works which are referred to in the text may be listed at the end of the paper under the heading "Bibliography" or "References." The following examples illustrate the two methods of arrangement and enumeration used in the Bulletin and Transactions:

Form I

BIBLIOGRAPHY [or REFERENCES]

1. R. R. Roe, *Canonical systems*, Amer. J. Math. vol. 60 (1938) pp. 1042–1100.
2. ———, *An elementary solution of the fundamental problem of mathematics*, J. of Metamath. vol. 17 (1939) pp. 1–100.
3. D. D. Doe, *The theory of topological hypergroups*, New York, 1940.

Form II

BIBLIOGRAPHY [or REFERENCES]

R. R. ROE
1. *Canonical systems*, Amer. J. Math. vol. 60 (1938) pp. 1042–1100.
2. *An elementary solution of the fundamental problem of mathematics*, J. of Metamath. vol. 17 (1939) pp. 1–100.

D. D. DOE
1. *The theory of topological hypergroups*, New York, 1940.
2. *A generalization of the generalized Doe-differential*, Duke Math. J. vol. 48 (1940) pp. 1100–1200.

The following examples illustrate the various methods of referring to the bibliography. The basic marks of enclosure are brackets: [ ]. With a small amount of care, the use of parentheses as additional marks of enclosure can be kept at a minimum and the appearance of the printed page thereby improved.

For form I: [1, Theorem 3.7]; · · · proved by Roe [2, Lemma 4] · · · ; (see [2, Lemma 4]).

For form II: [Roe 1, Theorem 3.7]; · · · proved by Roe [2, p. 100] · · · ; (cf. Roe [2, p. 100]).

The use of abbreviations for the names of the more obscure scientific journals often creates difficulties for readers and librarians. The Bulletin and Transactions, therefore, make it a rule to print such

titles without any abbreviations. *Cooperation on the part of the authors would diminish greatly the work involved in this connection.* The names of the standard journals, however, should be abbreviated. Appendix B contains a list of abbreviations which have been adopted by the Bulletin and Transactions.

Preparation of the manuscript. A compositor is not a trained mathematician and cannot take the responsibility of altering so much as the position of a single dot in the manuscript. It is obvious, therefore, that the manuscript, when it is sent to the printer, must be exact in every detail. Since it is impossible to produce a manuscript in which all characters are facsimiles of printed characters, there will be many details which can be made clear to the printer only by special indication such as underlining or marginal notes.

The task of preparing the manuscript for the printer is generally shared between author and editorial staff. In the case of the Bulletin and Transactions, indications concerning routine matters of style, such as the spacing of material, the choice of type in titles, paragraph headings, and so on, are inserted by an editorial assistant. All indications of a non-routine nature must be made by the author and must be complete and explicit. The most important rules in this connection are the following:[3]

1. The manuscript should be typewritten. The text should be double spaced and displayed material should not be crowded. Plenty of space should be left for the characters which are to be filled in by hand. It is well to make at least one carbon copy of the manuscript, but the *original*—not the carbon copy—should be sent to the editor.

2. Subscripts or superscripts may be filled in by hand; if typed they should be a half-space above or below the line.

3. Handwritten characters should be made with great care and each character must be distinct in shape from every other. In cases where the author's intention may be in doubt, identification, such as "Greek chi," should be written in pencil in the margin. Capital letters may be indicated by "cap," small letters by "l.c." (lower case). Letters and symbols which are particularly liable to confusion are:

$$o O 0 \quad \gamma r n v \quad \rho p \quad s S 5 \quad {}_1, \quad {}^1{}' \quad h k \kappa K \mathrm{K}$$
$$z Z 2 \quad \psi \chi x \times 4 \quad e l 1 \quad V \cup \quad \in \epsilon$$

4. German and script letters may be handwritten if they can be made with sufficient accuracy. A far more satisfactory procedure is to type the corresponding Roman letters, then to underline or encircle

[3] See also rules 3-8 in the section Special Rules.

each letter with colored pencil, say green for German, blue for script. In fact, all alphabets other than the Roman and Greek are best treated in this manner. The meaning of the colored marks should be stated at appropriate places in the margin or on a separate sheet of "instructions for the printer."

5. No mark meaning "Greek alphabet" should be attached to the symbols $\sum$ (summation), $\prod$ (product), ∂ (in partial derivatives). Notice the difference in the size and style between the first two symbols and the Greek capital letters Σ, Π.

6. Italic type is indicated by underlining, preferably with a blue pencil. In the Bulletin and Transactions, the text of a theorem or corollary is always in italics and the necessary indications may be left to an editorial assistant.

7. Omissions should be denoted by *exactly* three slightly raised dots, such as $ab \cdot \cdot \cdot c$. If commas are required, write $a_1, \cdot \cdot \cdot, a_n$ or $a_1, a_2, \cdot \cdot \cdot$, never $a_1, \cdot \cdot \cdot a_n$ or $a_1, a_2 \cdot \cdot \cdot$. A handwritten comma must be carefully made so that it will not be taken for subscript 1.

8. The number one is ordinarily typed 1. The letter "ell" is denoted by l (the dash may be written). Experience shows that the use of the solidus (/) in place of the dash may lead to errors in typesetting.

9. Drawings and diagrams should be made with particular care since they cannot be changed or corrected in galley proofs or page proofs. Drawings should be made in black India ink on Bristol board and should be two or three times larger than they are to be when printed. When inserting letters or symbols, allowance should be made for the reduction in size. For drawings which are to appear in the Bulletin or Transactions, all letters and symbols should be drawn lightly in pencil since they will eventually be replaced by type letters and symbols. The reproduction of even the simplest drawing involves considerable extra expense, and the author, therefore, should include no drawing which is not really essential for a clear understanding of his paper.

10. A displayed formula should be numbered only if referred to elsewhere in the text. If displayed formulas are numbered indiscriminately, it is impossible to combine several formulas into a single line or to run an occasional formula into the text to facilitate the breaking up of galleys into pages.

Proof reading. It is important that galley proofs be carefully read and corrected by the author, since it is only the author who can detect errors which are due to an imperfect manuscript.

Alterations in type which has been set are delicate and costly to

make. The author should call for only such changes as are really essential. The insertion or deletion of a comma, for example, should be based on a compelling reason and should not be merely a matter of whim. If a number of consecutive words or symbols must be deleted, they should if possible be replaced by words or symbols occupying the same amount of space. If this is not done, the compositor often must tear down and reset a number of preceding or following lines in order to make the necessary adjustments. This is not only costly but makes possible the introduction of new errors.

Galley proofs of papers for the Bulletin or Transactions are read in the office of the American Mathematical Society before they are sent to the author. During the reading, there may arise questions which can be answered only by the author. Such questions are written in the margins, and it would seem unnecessary to point out that they should be answered were it not for the fact that they are frequently ignored.

The list in Appendix A contains most of the standard signs used in this country in correcting proof. A consistent use of these signs by authors would simplify greatly the work of printer and editorial staff. In any case, authors should make sure that their corrections are clear and explicit and should refrain from using elaborate "systems" of their own invention. Joint authors should be particularly careful not to use two non-equivalent systems of signs in a single set of galleys.

The example in Appendix A illustrates the standard method of using the proof reading signs. It is important to note that all indications for corrections must appear in margins and for each correction a corresponding mark must appear in the text. The signs in the margins should retain the order of the corresponding errors in the line and should be separated from each other by |.

APPENDIX A

SIGNS USED IN CORRECTING PROOFS

- Delete; take out
- Close up
- ^ Insert
- # Insert space
- ⌐ Raise
- ⌙ Lower
- [Move to left
-] Move to right
- || Straighten type line at side of page
- // Straighten lines
- ¶ Paragraph
- center Put in middle of page or line
- ⤳ Transpose
- tr Transpose
- ↄ Turn inverted letter right side up
- x Change broken letter
- stet Let it stand as set
- ····· Let it stand as set
- w.f. Wrong font, size or style
- l.c. Lower case, not capitals
- rom. Use Roman letter
- bf. Use black type letters
- ⊙ Period
- ⋀ Comma
- ⋎ Apostrophe
- ⋎ 2 Superior figure
- ⋀ 2 Inferior figure
- =/ Hyphen
- sc. Use small capitals
- caps Use capitals
- ital. Use italics

EXAMPLE SHOWING THE USE OF PROOF READING SIGNS

of B_1 and d_1d_2' divides d_1d_2' (+1). Hence B is reducible to the form (11.5) with diagonal terms $d_1, d_1d_2', \cdots, d_1d_n'$ which proved (11.4).

12. **Groups with a finite number of generators.** We shall discuss certain properties of these groups culminating in the basic product decomposition (12.5).

(12.1) DEFINITION. *Let* $B=\{g_1, \cdots, g_n\}$, $B'=\{g_1', \cdots, g_n'\}$ *be two sets of elements of* G *containing the same number* n *of elements. By a unimodular transformation* $\tau: B \to B'$ *is meant a system of relations*

$$g_i' = \sum a_{ij}g_j, \qquad \|a_{ij}\| \text{ unimodular.} \tag{12.2}$$

The following proposition shows in how natural a manner unimodular transformations make their appearance in the theory of groups with finite bases.

(12.3) *Let* G *be a group with a finite base* $B=\{g_1, \cdots, g_n\}$. *In order that* $B'=\{g_1', \cdots, g_n'\}$ *be a base for* G *it is necessary and sufficient that* B' *be obtainable from* B *by a unimodular transformation.*

For any given set $B'=\{g_1', \cdots, g_n'\}$ of elements of G there exist relations

$$g_i' = \sum c_{ij}g_j, \qquad C=\|c_{ij}\| \tag{12.4}$$

A necessary and sufficient condition ~~in order~~ that $\{g_i'\}$ be a base is that the g_j be expressible as linear combinations of the g_i', or that there exist relations

$$g_i = \sum d_{ij}g_j', \qquad D=\|d_{ij}\|. \tag{12.5}$$

From this follows

$$g_i = \sum d_{ij}c_{jk}g_k,$$

Hence since B is a base we must have $DC=1$.)

This matrix relation yields $|D|\,|C|=1$, and since the determinants are integers we must have $|C|=\pm 1$. Thus in order that B' be a base C must be unimodular, or the condition of (12.3) must be fulfilled. Conversely, if (12.2) is fulfilled, C is unimodular and (12.5) holds with $D=C(-1)$, from which it follows readily that B' is a base.

THE PRECEDING PASSAGE PRINTED WITH ALL CORRECTIONS MADE

of B_1 and $d_1 d_p'$ divides $d_1 d_{p+1}'$. Hence B is reducible to the form (11.5) with diagonal terms d_1, $d_1 d_2'$, $\cdots$, $d_1 d_n'$ which proves (11.4).

12. **Groups with a finite number of generators.** We shall discuss certain properties of these groups culminating in the basic product decomposition (12.5).

(12.1) DEFINITION. *Let* $B = \{g_1, \cdots, g_n\}$, $B' = \{g_1', \cdots, g_n'\}$ *be two sets of elements of* G *containing the same number* n *of elements. By a unimodular transformation* $\tau: B \to B'$ *is meant a system of relations*

$$g_i' = \sum a_{ij} g_j, \qquad \|a_{ij}\| \text{ unimodular.} \tag{12.2}$$

The following proposition shows in how natural a manner unimodular transformations make their appearance in the theory of groups with finite bases.

(12.3) *Let* G *be a group with a finite base* $B = \{g_1, \cdots, g_n\}$. *In order that* $B' = \{g_1', \cdots, g_n'\}$ *be a base for* G *it is necessary and sufficient that* B' *be obtainable from* B *by a unimodular transformation.*

For any given set $B' = \{g_1', \cdots, g_n'\}$ of elements of G there exist relations.

$$g_i' = \sum c_{ij} g_j, \qquad C = \|c_{ij}\|. \tag{12.4}$$

A necessary and sufficient condition that $B' = \{g_i'\}$ be a base is that the g_j be expressible as linear combinations of the g_i', or that there exist relations

$$g_i = \sum d_{ij} g_j', \qquad D = \|d_{ij}\|. \tag{12.5}$$

From this follows

$$g_i = \sum d_{ij} c_{jk} g_k.$$

Hence since B is a base we must have $DC = 1$. This matrix relation yields $|D| \cdot |C| = 1$, and since the determinants are integers we must have $|C| = \pm 1$. Thus in order that B' be a base C must be unimodular, or the condition of (12.3) must be fulfilled. Conversely, if (12.3) is fulfilled, C is unimodular and (12.5) holds with $D = C^{-1}$, from which it follows readily that B' is a base.

APPENDIX B

ABBREVIATIONS USED IN THE BULLETIN AND TRANSACTIONS

Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität (formerly . . . Hamburgischen Universität)	Abh. Math. Sem. Hansischen Univ.
Academy of Sciences of the USSR. Journal of Physics	Acad. Sci. USSR. J. Phys.
Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum	Acta Univ. Szeged.
Acta Mathematica	Acta Math.
American Journal of Mathematics	Amer. J. Math.
The American Mathematical Monthly	Amer. Math. Monthly
American Mathematical Society Colloquium Publications	Amer. Math. Soc. Colloquium Publications
American Statistical Association Bulletin	Amer. Statist. Assoc. Bull.
Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques	Ann. Fac. Sci. Univ. Toulouse
Annales Scientifiques de l'École Normale Supérieure	Ann. École Norm.
The Annals of Mathematical Statistics	Ann. Math. Statist.
Annals of Mathematics	Ann. of Math.
Atti della Reale Accademia delle Scienze di Torino. Classe di Scienze fisiche, Matematiche e Naturali	Atti Accad. Sci. Torino
Il Bollettino di Matematica	Boll. Mat.
Bulletin de l'Académie des Sciences de l'URSS. Série Mathématique	Bull. Acad. Sci. URSS. Sér. Math.
Bulletin de la Société Mathématique de France	Bull. Soc. Math. France
Bulletin de la Société Physico-Mathématique de Kazan et de l'Institut de Mathématiques et Mécanique	Bull. Soc. Phys.-Math. Kazan
Bulletin des Sciences Mathématiques	Bull. Sci. Math.
Bulletin Mathématiques de l'Université de Moscou. Série Internationale	Bull. Math. Univ. Moscou
Bulletin of the American Mathematical Society	Bull. Amer. Math. Soc.
Bulletin of the Calcutta Mathematical Society	Bull. Calcutta Math. Soc.
Commentarii Mathematici Helvetici	Comment. Math. Helv.
Compositio Mathematica	Compositio Math.
Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS	C. R. (Doklady) Acad. Sci. URSS.
Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences	C. R. Acad. Sci. Paris
Doklady Akademii Nauk SSSR See Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS	
Duke Mathematical Journal	Duke Math. J.
Fundamenta Mathematicae	Fund. Math.
Izvestia Akademii Nauk SSSR See Bulletin de l'Académie des Sciences de l'URSS	

Jahresbericht der Deutschen Mathematiker Vereinigung	Jber. Deutschen Math. Verein.
Japanese Journal of Mathematics. Transactions and Abstracts	Jap. J. Math.
Journal de l'École Polytechnique	J. École Polytech.
Journal de Mathématiques Pures et Appliquées	J. Math. Pures Appl.
Journal für die reine und angewandte Mathematik	J. Reine Angew. Math.
The Journal of Symbolic Logic	J. Symbolic Logic
Journal of the Faculty of Science. Imperial University of Tokyo. Section I. Mathematics, Astronomy, Physics, Chemistry	J. Fac. Sci. Imp. Univ. Tokyo. Sect. I
The Journal of the Indian Mathematical Society	J. Indian Math. Soc.
The Journal of the London Mathematical Society	J. London Math. Soc.
Koninklijke Nederlandsche Akademie van Wetenschappen. Indagationes Mathematicae ex Actis Quibus Titulus. Proceedings of the Section of Sciences (sometimes referred to as the "Amsterdam Proceedings")	Neder. Akad.. Wetensch.
Matematičeskij Sbornik See Recueil Mathématique	
Mathematische Annalen	Math. Ann.
Mathematische Zeitschrift	Math. Zeit.
Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. Neue Folge. Fachgruppe I. Nachrichten aus der Mathematik	Nachr. Ges. Wiss. Göttingen
Periodico di Matematiche	Period. Math.
Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences	Philos. Trans. Roy. Soc. London. Ser. A.
Proceedings of the Cambridge Philosophical Society	Proc. Cambridge Philos. Soc.
Proceedings of the Imperial Academy	Proc. Imp. Acad. Tokyo
Proceedings of the London Mathematical Society	Proc. London Math. Soc.
Proceedings of the National Academy of Sciences of the United States of America	Proc. Nat. Acad. Sci. U.S.A.
Proceedings of the Royal Society. Series A. Mathematical and Physical Sciences	Proc. Roy. Soc. London. Ser. A.
The Quarterly Journal of Mathematics Oxford Series	Quart. J. Math. Oxford Ser.
Recueil Mathématique. Nouvelle Série	Rec. Math. (Mat. Sbornik) N.S.
Rendiconti del Circolo Matematico di Palermo	Rend. Circ. Mat. Palermo
Sitzungsberichte der Berliner Mathematischen Gesellschaft	Berlin Math. Ges. Sitzungsber.
Sitzungsberichte der Preussischen Akademie der Wissenschaften	Preuss. Akad. Wiss. Sitzungsber.
The Tôhoku Mathematical Journal	Tôhoku Math. J.
Transactions of the American Mathematical Society	Trans. Amer. Math. Soc.

APPENDIX C

A LIST OF SIGNS AND SPECIAL CHARACTERS AVAILABLE FOR THE BULLETIN AND TRANSACTIONS

Lightface Greek—α β γ . . . (all) A B Γ . . . (all).

★ Lightface Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma}$ . . . (all except o).

★ Lightface Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma}$ . . . (all except o).

* Boldface Greek—α β γ Δ δ ε ζ η θ κ λ μ ν ξ Π π ρ σ τ Φ φ ψ Ω ω.

* Lightface German—𝔄 𝔅 ℭ . . . (all) 𝔞 𝔟 𝔠 . . . (all).

* Lightface dashed German—$\bar{\mathfrak{a}}$ $\bar{\mathfrak{B}}$ $\bar{\mathfrak{C}}$ $\bar{\mathfrak{I}}$ $\bar{\mathfrak{K}}$ $\bar{\mathfrak{M}}$ $\bar{\mathfrak{m}}$ $\bar{\mathfrak{n}}$ $\bar{\mathfrak{o}}$ $\bar{\mathfrak{p}}$ $\bar{\mathfrak{q}}$ $\bar{\mathfrak{R}}$ $\bar{\mathfrak{r}}$ $\bar{\mathfrak{s}}$ $\bar{\mathfrak{S}}$ $\bar{\mathfrak{U}}$ $\bar{\mathfrak{u}}$ $\bar{\mathfrak{v}}$ $\bar{\mathfrak{w}}$

* Boldface German—𝖆 𝕬 𝕭 𝕯

Script (special font) $\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$ . . . (all). No lower case manufactured.

* Hebrew—ℵ ℶ ℸ ℷ troublesome to use. No superior or inferior.

★ Dashed Italics—$\bar{A}$ $\bar{a}$ $\bar{B}$ $\bar{b}$ $\bar{C}$ $\bar{c}$ . . . (all)

★ Tilde Italics—$\tilde{A}$ $\tilde{a}$ $\tilde{B}$ $\tilde{C}$ $\tilde{D}$ $\tilde{E}$ $\tilde{e}$ $\tilde{g}$ $\tilde{H}$ $\tilde{I}$ $\tilde{J}$ $\tilde{K}$ $\tilde{M}$ $\tilde{N}$ $\tilde{n}$ $\tilde{O}$ $\tilde{r}$ $\tilde{S}$ $\tilde{s}$ $\tilde{T}$ $\tilde{u}$ $\tilde{y}$ $\tilde{W}$ $\tilde{Z}$

★ Tilde Greek— $\tilde{\alpha}$ $\tilde{\Gamma}$ $\tilde{\gamma}$ $\tilde{\varepsilon}$ $\tilde{\eta}$ $\tilde{\theta}$ $\tilde{\iota}$ $\tilde{\lambda}$ $\tilde{o}$ $\tilde{\upsilon}$ $\tilde{\omega}$

★ Dashed Greek—$\bar{\alpha}$ $\bar{\beta}$ $\bar{\Gamma}$ $\bar{\gamma}$ $\bar{\delta}$ $\bar{\varepsilon}$ $\bar{\zeta}$ $\bar{\eta}$ $\bar{\theta}$ $\bar{\iota}$ $\bar{\kappa}$ $\bar{\lambda}$ $\bar{\mu}$ $\bar{\nu}$ $\bar{\xi}$ $\bar{\pi}$ $\bar{\rho}$ $\bar{\sigma}$ $\bar{\phi}$ $\bar{\psi}$ $\bar{\omega}$

★ Dotted Italic—$\dot{a}$ $\ddot{a}$ $\dot{e}$ $\ddot{e}$ $\ddot{g}$ $\ddot{\imath}$ $\dot{m}$ $\dot{n}$ $\dot{q}$ $\dot{r}$ $\ddot{r}$ $\dot{s}$ $\dot{u}$ $\ddot{u}$ $\dot{x}$ $\ddot{x}$ $\dot{y}$ $\ddot{y}$ $\dot{z}$ $\ddot{z}$

★ Dotted Greek—$\ddot{\varepsilon}$ $\dot{\eta}$ $\ddot{\eta}$ $\dot{\theta}$ $\ddot{\theta}$ $\dot{\xi}$ $\ddot{\iota}$ $\dot{\psi}$ $\ddot{\psi}$ $\ddot{\omega}$ (single dotted β γ δ φ; double dotted γ readily available).

★ Superiors—^{m} ^{r} ^{s} ^{1} $^{\pm}$

★ Inferiors—$_{m}$ $_{n}$ $_{a}$ $_{r}$ $_{s}$ $_{\pm}$ $_{y}$ $_{z}$ $_{i}$ $_{\pm}$

Signs—Superior—$^{-}$ $^{+}$ $^{=}$ $^{\infty}$ $^{\pm}$ $^{(}$ $^{)}$ $^{[}$ $^{]}$ $^{.}$ $^{.}$ $^{'}$ $^{/}$ $^{|}$

Signs—Superior to Superior—$^{+}$ $^{-}$ $^{-}$

Signs—Inferior—$_{-}$ $_{+}$ $_{=}$ $_{\infty}$ $_{\pm}$ $_{\neq}$ $_{(}$ $_{)}$ $_{[}$ $_{]}$ $_{.}$ $_{.}$ $_{/}$ $_{|}$ $_{'}$

Signs—Inferior to Inferior—$_{+}$ $_{-}$ $_{-}$

Special—Superior to Superior—^{1} ^{2} ^{3} ^{0} ^{m}

Special—Inferior to Inferior—$_{1}$ $_{2}$ $_{3}$ $_{4}$ $_{r}$ $_{s}$ $_{i}$ $_{j}$ $_{k}$ $_{l}$ $_{m}$ $_{n}$ $_{p}$ $_{r}$ $_{s}$

* Additional characters readily available at small cost.

★ Matrices for additional characters are made upon special orders. Normally they would necessitate a delay of from four to eight weeks and average expense of $6.00 per matrix. However, because of present-day manufacturing conditions delivery dates are uncertain and prices subject to change without notice.

No.	6	8	10	12	No.	6	8	10	12[4]
1. −	√	√	√	√	40. ∨		√	√	
2. ×	√	√	√	√	41. ⊃			√	
3. ⋇		√			42. ⊂			√	
4. ÷	√	√	√	√	43. ⊅			√	
5. +	√	√	√	√	44. ∪	√			
6. ┼			√		45. ∩		√		
7. ≠		√			46. ∴			√	
8. ±	√	√	√	√	47. △			√	
9. ∓			√		48. ▲			√	
10. =	√	√	√	√	49. ◬		√		
11. 〓		√	√		50. ▲		√		
12. ≐	√		√		51. ∠		√	√	
13. ≅	√		√		52. ◿			√	
14. ≦		√			53. □			√	
15. ≠	√	√	√		54. ⊡		√		
16. ≠		√	√		55. ■		√		
17. ≡		√	√		56. ▭			√	
18. ≢		√	√		57. $			√	
19. ≢			√		58. ▱			√	
20. ∞	√	√	√		59. $			√	
21. ∽		√	√		60. ◇		√		
22. ∝			√		61. ○		√	√	
23. ≎		√	√		62. ⊙		√	√	
24. ∼		√	√		63. ⊜			√	
25. ∽		√			64. ⊙		√		
26. ≈			√		65. ⊙		√		
27. <	√	√	√		66. ⊖			√	
28. ≤			√		67. ⊕			√	
29. >	√	√	√		68. ⊗		√	√	
30. ≷			√		69. ⊘			√	
31. ≦	√	√	√		70. ⊠			√	
32. ≧	√	√	√		71. ▽			√	
33. ≯		√	√	√	72. →	√	√	√	
34. ≮		√			73. ←		√	√	
35. ∧		√	√		74. →	√	√	√	
36. ∨		√	√		75. ⇄			√	
37. ⊼		√			76. ⇆				√
38. ≪			√		77. ↶		√		
39. ≮			√		78. ⇄			√	

[4] These figures refer to the size of type. The text of Bulletin and Transactions is 10 point type and footnotes 8 point type.

No.	6	8	10	12	No.	6	8	10	12
79. ⇆			√		132. ☆		√		√
80. ↗			√		133. ★	√	√		√
81. ↘	√		√		134. ❖	√			
82. /	√	√	√	√	135. Ⓐ		√		
83. \		√	√		136. ●	√			
84. ⫞			√		137. Ⓟ		√		
85. —	√	√	√	√	138. Ⓡ		√		
86. ‖	√	√	√	√	139. ϒ	√		√	
87. ⟋		√	√		140. ⩽			√	
88. ⟍		√	√		141. ⁿ		√		
89. ⫽			√		142. ᵖ	√			
90. ⫻			√		143. ⁀		√		
91. √	√	√	√	√	144. ◡	√			
100. ⊥		√	√		145. ˜		√		
101. ⊢			√		146. ˌ	√			
102. ⊨			√		157. ⊂			√	
103. ⫤		√			158. ⊃			√	
104. ⫣			√		159. ⊆			√	
105. ⫯			√		160. ⊇			√	
106. ′	√	√	√	√	161. ⊄			√	
107. ‵			√		162. ∈			√	
108. ″	√		√		163. ⊃			√	
109. ‧	√				164. ⊂			√	
110. ⁝		√			165. ∉			√	
111. ⁞		√	√		166. ∪				√
112. ∫			√		167. ↔			√	
113. ∫	√	√	√	√	168. ⫢			√	
114. ∫		√	√		169. ⫡			√	
115. ∏			√	√	170. #	√	√	√	
116. ∑			√	√	171.				
117. °	√	√	√	√	172.				
118. ⁰		√	√		173.				
119. ·		√	√		174.				
120. •		√	/		175.				
121. :		√	√		176.				
122. ⋮			√		177.				
129. ⟨			√		178.				
130. ⟩			√		179.				
131. ✯			√		181.				